H_2 Control of Systems with I/O Delays

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DISC

Dutch Institute of Systems and Control

Cover picture: a digitally manipulated version of a picture taken at The Jakarta International Airport. All flight numbers, and therefore all flights, in the picture are fictitious. No actual flights were delayed.

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DISSERTATION

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Acronyms

ARE   algebraic Riccati equation
FIR   finite impulse response
LFT   linear fractional transformation
LQG   linear quadratic Gaussian
LQR   linear quadratic regulator
LTI   linear time invariant
MIMO multi input multi output
SISO   single input single output

x

Notation

$\mathbb{R}, \mathbb{C}$	real numbers and complex numbers
$\mathbb{R}^{m \times n}$	real $m \times n$ matrices
$j\mathbb{R}$	imaginary axis
$\mathbb{1}(t)$	unit step function
$\delta(t)$	unit impulse function
$F(t), F(s)$	impulse response and transfer function of the system F
I	identity matrix
$\mathrm{col}[a_1, \ldots, a_n]$	a column vector with components $a_1, \ldots, a_n$
$\mathrm{diag}(a_1, \ldots, a_n)$	an $n \times n$ diagonal matrix with a_i as its i-th diagonal element
A^T	transpose of A
A^{-1}	inverse of A
$\det(A)$	determinant of A
$\mathrm{trace}(A)$	trace of A
$\bar{\sigma}(A)$	largest singular value of A
$L_2(-\infty, \infty)$	time domain square integrable functions
$L_2[0, \infty)$	subspace of $L_2(-\infty, \infty)$ with functions zero for $t < 0$
$L_2(-\infty, 0]$	subspace of $L_2(-\infty, \infty)$ with functions zero for $t > 0$
$L_2(j\mathbb{R})$	square integrable functions on $j\mathbb{R}$
H_2	subspace of $L_2(j\mathbb{R})$ with functions analytic in $\mathrm{Re}(s) > 0$
$H_2^\perp$	subspace of $L_2(j\mathbb{R})$ with functions analytic in $\mathrm{Re}(s) < 0$
$L_\infty(j\mathbb{R})$	functions bounded on $\mathrm{Re}(s) = 0$ including at ∞
H_∞	the set of $L_\infty(j\mathbb{R})$ functions analytic in $\mathrm{Re}(s) > 0$
$D^\perp$	orthogonal complement of D
$\langle, \rangle$	inner product
$F^\sim(s)$	$F^T(-s)$
$\phi_h(F(s))$	completion of the transfer function $F(s)$
$\tau_h(F(s))$	truncation of the transfer function $F(s)$
$\{F(s)\}_+$	stable part of the transfer function $F(s)$
$S_\mathrm{u}(F(s))$	upper Schur transformation of $F(s)$
$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$	shorthand for $C(sI - A)^{-1}B + D$
$\mathrm{Re}(\alpha)$	real part of $\alpha \in \mathbb{C}$
$g(t) * f(t)$	$\int_{-\infty}^{\infty} g(t - \tau)f(\tau)d\tau$
$F_\ell(P, K)$	lower linear fractional transformation

$F_u(P, K)$ upper linear fractional transformation
$C_\ell(G, K)$ left homographic transformation

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Introduction and Preliminaries

1.1. Motivation

Time delay is a phenomenon that arises in many physical systems. Among the major causes are transportation and measurement lags, analysis times, computation times, and communication lags [Pal96]. Examples of mathematical models describing physical systems that involve time delays are numerous. The book [KM99] gives delay system examples in mechanics, physics, engineering, biology, medicine, and economy. An interesting example is the use of a time delay system in the modeling of population growth, which is described in what follows.

Example 1.1 (Single species population growth [KM99]). The following equation is a simple model of single species population growth with limited self-renewing food resources:

$$\dot{x}(t) = \gamma[1 - \frac{1}{K}x(t-h)]x(t).$$

Here, $x(t)$ is the population size at time t. The constant γ is related to the reproduction rate of the species, i.e. the difference between the birth rate and the death rate. The constant K is the average population size and is related to the ability of the environment to sustain the population. The constant h is the production time of food resources. It enters the equation as a delay because food resources at time t, which affect the population growth at time t, are determined by the population size at time $t - h$.

Time delay may also be used as a means of model reduction for high order or infinite dimensional systems. For example, it is shown in [ZB97] that a heated can system described by a partial differential equation may be well-approximated by a low order system with a delay.

The presence of delays in a control system poses a challenge to controller design. In addition to complicating system analysis and controller design, they also make satisfactory control more difficult to achieve [Pal96].
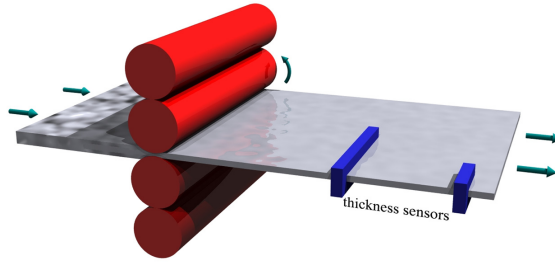
Figure 1.1.: A steel rolling mill.

The fact that delay is a naturally occurring physical phenomenon and that it is not covered by the well-established finite dimensional systems and control theory makes delay systems a very active research area. The survey paper [Ric03] notes that at least thirty English language books have been written since 1963 in the area of time delay systems. The results in this area are indeed extensive and cover a very broad aspects of time delay systems. For a general overview of time delay systems, we refer to the books [KM99], [GKC03] and the survey paper [Ric03]. This thesis deals with a class of delay systems where time delays occur only in the input/output channels of the (generalized) linear time invariant plant[1]. The following example describes an industrial system that exhibits output delays.

Example 1.2 (Steel rolling mill [GH98]). Figure 1.1 shows an example of a system with multiple output delays: a steel rolling mill. The control problem associated with the rolling mill is to regulate the cross section profile of the outgoing steel sheet in the presence of temperature and thickness non-uniformity in the incoming steel sheet. It is done by shaping the gap between the rollers by applying forces that bend and press the rollers. The controller computes the forces based on two thickness measurements at the edge and at the center of the outgoing steel sheet. Due to certain circumstances, the two thickness sensors have to be placed at different distances from the rollers, resulting in different delays affecting the two measurements. The rolling mill control system will be discussed in greater detail in Chapter 6.

The first attempt to incorporate delays in the plant model and subsequently design a controller based on the model was perhaps done by Smith [Smi57], resulting in the well-known Smith predictor. Smith transformed the stabilization problem of stable SISO plants with a delay to a finite dimensional stabilization problem, to which classical control design techniques

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[1]For example, this class does not cover all cases where delays occur in the state.
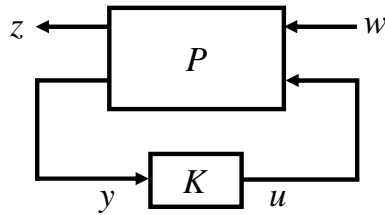
Figure 1.2.: The LFT (linear fractional transformation) general framework.

may be applied. The Smith predictor has been extended to cover unstable plants and MIMO plants with multiple delays (see [Pal96]).

As control theory is shifting from the classical control paradigm to the modern control paradigm, it is natural to extend modern control techniques for finite dimensional systems to delay systems. The research in this area was pioneered by Kleinman [Kle69] who solved the LQG problem for systems with a single delay. This work was followed by more results in H_2 synthesis ([DM72a], [DM72b], [DMM75], [PS85], [PS87], [Del86], [MR03]) and H_∞ synthesis ([vK93], [FÖT96], [MZ00], [KI03a], [Mir03a], [MM05a]) for time delay systems.

Although a lot of results have been put forward, most of the results in H_2 synthesis only cover the single delay case. The multiple delays case, where different delays may occur in each channel of the input and output, is largely untouched[2]. Thus, control systems with multiple i/o delays such as the rolling mill system in Example 1.2 cannot be treated. This fact motivates the research effort that resulted in this thesis. The aim is to develop a solution of the standard H_2 control problem for systems with multiple i/o delays. The effort resulted in two solutions of the problem: a time domain solution and a frequency domain solution.

1.2. About this thesis

This thesis primarily deals with the standard H_2-optimal control problem of systems with multiple input/output delays. Two solutions are developed: a frequency domain solution and a time domain solution. The development of the two solutions gives rise to the solution of two closely related problems, namely the parametrization of all stabilizing controllers for systems with multiple i/o delays and the H_2 control problem of preview systems. The former is an important preliminary for the frequency domain approach,

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[2]Notable exceptions are the papers [SR72] and [KI03a] that treat the LQR problem with multiple input delays. However, the results do not cover the output feedback H_2 problem for systems with multiple i/o delays.

while the solution of the latter is inspired by the time domain approach.

The LFT (linear fractional transformation), shown in Figure 1.2, is used as the framework for controller design in this thesis. In this framework, P is the generalized plant and K is the controller. The signals w and u are called the external input and the internal input, respectively, while the signals z and y are called the controlled output and measurement output, respectively. The external input w contains all inputs that cannot be manipulated. This includes disturbances, noise, and reference signals. The control input u contains all inputs that can be manipulated. The measurement output y contains measurements that are fed to the controller. Finally, the controlled output z contains variables that we would like to control, such as error signals.

For systems with multiple i/o delays, the control input u and the measurement y may be delayed before being fed to the plant and the controller, respectively. Furthermore, each channel of the multi-valued signals u and y may have different time delays. Preview systems are systems where all or part of the external input w is known in advance. In this case, the measurement signal y contains two components: the state x and a time-advanced version of the external input w, where each channel of the multi-valued signal w may have different preview time.

In both systems with delays and preview systems cases, the aim is to find the controller that minimizes the transfer function from the external input to the controlled output in the H_2 sense. The idea is to make the mapping from w to z small in some sense, so that the effect of the external signal w to the controlled output z also becomes small. In the H_2-optimization case, the optimal controller may be interpreted as the controller that minimizes the energy of z when w is a series of unit impulses. It has also a stochastic interpretation of minimizing the expected power of z when w is a unit intensity white noise input. The latter interpretation makes H_2 control design equivalent to the LQG (linear quadratic Gaussian) control design. In fact, it may be seen as a deterministic equivalence of the stochastic optimization in the LQG framework.

The thesis consists of seven chapters. The content of the chapters is outlined in what follows.

Chapter 1 This chapter provides a brief introduction to the topics discussed in this thesis, along with some necessary preliminaries.

Chapter 2 This chapter presents the parametrization of all stabilizing controllers for systems with multiple i/o delays. The result is obtained by transforming the stabilization setup to an equivalent finite dimensional setup. This result is proved useful in deriving a frequency domain solution of the standard H_2 control problem of systems with multiple i/o delays, which is derived in Chapter 4. The results in this chapter are based on the paper

[MM03].

Chapter 3 In this chapter, a time domain solution to the standard H_2 control problem of systems with multiple i/o delays is derived. In this approach, the standard problem is reduced to solving a linear quadratic regulator (LQR) problem with multiple input delays, which is then solved by breaking it into a finite number of standard delay-free LQR problems. Although the method only covers the case where the delays occur solely on one side of the controller, the resulting formulas are straightforward and easy to implement. This chapter is based on the papers [MM04] and [MM05c].

Chapter 4 This chapter discusses a frequency domain solution to the standard H_2 control problem of systems with multiple i/o delays. The solution is obtained by first reducing the standard problem to what is called two-sided regulator problem. The result from Chapter 2 plays an important role in the transformation. The two-sided regulator problem is then solved using spectral factorization arguments. The frequency domain solution covers the general case where delays occur on both sides of the controller. However, the method is more complicated and less elegant than the time domain solution and cannot handle unstable plants. This chapter is based on the papers [MMK03] and [MMK05].

Chapter 5 This chapter deals with the H_2 control problem of systems with preview, where the knowledge of all or part of the external input is known some time in advance. It turns out that the time domain techniques developed in Chapter 3 for solving the H_2 control problem for systems with i/o delays may easily be adapted to solve the problem for systems with preview. This chapter is based on the paper [MM05b].

Chapter 6 The aim of this chapter is to demonstrate the potential of the theory developed in the preceding chapters for application. This objective is achieved by presenting two case studies: control of a steel rolling mill (a delay system) and control of a container crane (a tracking system with preview). In addition, the practical implementation of finite impulse response systems, an important part of optimal controllers devised in this thesis, is also discussed.

Chapter 7 This chapter presents the conclusion and a discussion about future research and open problems.

1.3. A review of linear systems theory

In this section some essential concepts and results from linear systems theory are outlined. The text, which is based on material from the books [ZD98], [GL95], and [SP96], only covers concepts and results that form a

necessary background for the subsequent chapters.

1.3.1. Signals and systems

A *signal* is formally defined as a Lebesgue measurable function that maps the real numbers $\mathbb{R}$ to $\mathbb{R}^n$. An important class of signals is the $L_2(-\infty, \infty)$ signal space which is defined by

$$L_2(-\infty, \infty) = \{f : \mathbb{R} \to \mathbb{R}^n, \|f\|_2 < \infty\}$$

where the associated norm is defined via

$$\|f\|_2^2 = \int_{-\infty}^{\infty} \|f(t)\|^2 dt.$$

Here $\|f(t)\| = \sqrt{f(t)^T f(t)}$ is the Euclidean norm. The subspaces $L_2[0, \infty)$ and $L_2(-\infty, 0]$ are the subspaces of $L_2(-\infty, \infty)$ signals that are identically zero for $t < 0$ and $t > 0$, respectively.

A *system* is a mapping from one signal space, the input space, to another signal space, the output space. A system F is called *linear* if for any scalars (α, β) and any inputs (u_1, u_2), the following holds:

$$F(\alpha u_1 + \beta u_2) = \alpha F u_1 + \beta F u_2.$$

A system F is called *time invariant* if the following holds:

$$y(t) = Fu(t) \Rightarrow y(t - T) = Fu(t - T), \quad \forall T.$$

1.3.2. System description

We discuss three representation of a system: state space representation, impulse response representation, and transfer matrix representation.

State space representation

Any system described by an ordinary linear differential equation with constant coefficients may be represented by the state space equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$
$$y(t) = Cx(t) + Du(t). \tag{1.2}$$

Here, x is the state with dimension n, u is the input with dimension p, and y is the output with dimension m. The matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times p}$ are real matrices. Given the system described by

(1.1,1.2) with an initial condition $x(t_0)$, the state trajectory may be computed using

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau,$$

where the matrix exponential

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

Two important system properties related to the state space representation are controllability and observability. Controllability basically means that the state of a system may be steered to any point by applying an appropriate input, while observability roughly means that the external signals (u, y) contain all information of the state and it is possible to obtain the state by measuring the output. The two concepts are precisely defined in what follows.

Definition 1.3 (Controllability). The dynamical system $\dot{x} = Ax + Bu$, or the pair (A, B), is said to be *controllable* if, for any initial state $x(0)$ and any final state x_1, there exists time $t_1 > 0$ and an input $u(t)$ such that $x(t_1) = x_1$. Otherwise the system or the pair is said to be *uncontrollable*.

One way to check controllability is from the rank the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}.$$

The pair (A, B) is controllable if and only if $\mathcal{C}$ has rank n.

Definition 1.4 (Observability). The dynamical system $\dot{x} = Ax + Bu, y = Cx + Du$, or the pair (C, A) is said to be *observable* if, there exists a time $t_1 > 0$ such that the initial state $x(0)$ can be uniquely determined from the history of the input $u(t)$ and the output $y(t)$ in the interval $[0, t_1]$. Otherwise the system or the pair is said to be *unobservable*.

Similar to the controllability case, observability may be determined from the rank of the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

The pair (C, A) is observable if and only if $\mathcal{O}$ has rank n.

Another two important concepts are stabilizability and detectability.

Definition 1.5 (Stabilizability). The dynamical system $\dot{x} = Ax + Bu$, or the pair (A, B), is said to be *stabilizable* if, there exists a sate feedback $u = Fx$ such that the system is stable, i.e. $(A + BF)$ is Hurwitz[3].

Definition 1.6 (Detectability). The dynamical system $\dot{x} = Ax + Bu, y = Cx + Du$, or the pair (C, A) is said to be *detectable* if, there exists a matrix L such that $(A + LC)$ is Hurwitz.

The following definitions of modal controllability and observability provide alternatives to definitions 1.3 and 1.4 [ZDG96].

Definition 1.7 (Modal controllability and observability). Consider the dynamical system $\dot{x} = Ax + Bu, y = Cx + Du$. Let λ be an eigenvalue of A or, equivalently, a *mode* of the system. Then the mode λ is said to be controllable (observable) if $x^T B \neq 0$ ($Cx \neq 0$) for all left (right) eigenvectors of A associated with λ; that is, $x^T A = \lambda x^T$ ($Ax = \lambda x$) and $x \neq 0$. Otherwise, the mode is said to be uncontrollable (unobservable).

It follows that a system is controllable (observable) if and only if every mode is controllable (observable). Similarly, a system is stabilizable (detectable) if and only if every unstable mode is controllable (observable).

Impulse response representation

Another way to describe a linear time invariant system (LTI) is through its impulse response. For any LTI system F with input u and output y, there exists a function $F(t)$ such that the input-output mapping is given by:

$$y(t) = F(t) * u(t) = \int_{-\infty}^{\infty} F(t - \tau)u(\tau)d\tau.$$

The function $F(t)$ is called the *impulse response* of the system F.

For SISO (single input single output) systems, the impulse response is the output of the system when the input is the unit impulse function $\delta(t)$ which satisfies

$$\lim_{\epsilon \downarrow 0} \int_{-\epsilon}^{\epsilon} \delta(t)dt = 1,$$

and $\delta(t) = 0$ for $t \neq 0$.

For MIMO (multi input multi output) system F with input u and output y, the (i, j)-th element of the impulse response matrix, denoted by $F_{ij}(t)$ is the response $y_i(t)$ to an impulse $u_j(t) = \delta(t)$. The impulse response of a system described by the state space representation (1.1,1.2) is defined by[4]

$$Ce^{At}B\mathbb{1}(t) + D\delta(t),$$

---

[3]A square matrix A is called *Hurwitz*, or equivalently *stable*, if all its eigenvalues have a strictly negative real part.

[4]We assume that the input, the output, and the state are initially at rest signals.

where $\mathbb{1}(t)$ is the unit step function defined as

$$\mathbb{1}(t) = \left\{ \begin{array}{ll} 1, & t \geq 0 \\ 0, & t < 0 \end{array} \right. .$$

Transfer matrix representation

The transfer matrix of a system F, which is denoted by $F(s)$, is defined as the (two-sided) Laplace transform of its impulse response:

$$F(s) = \int_{-\infty}^{\infty} F(t)e^{-st}dt. \tag{1.3}$$

The input-output mapping is given by

$$y(s) = F(s)u(s),$$

where $y(s)$ and $u(s)$ are the Laplace transform of the output $y(t)$ and the input $u(t)$, respectively. The transfer matrix of a system described by the state space representation (1.1,1.2) is given by[5]

$$C(sI - A)^{-1}B + D =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right], \tag{1.4}$$

with the region of convergence being $\mathrm{Re}(s) > \alpha$, where α is the real part of the pole of $C(sI - A)^{-1}B + D$ that is located farthest to the right in the complex plane. The righthand side of the above equation is a shorthand notation that will be often used in this thesis.

1.3.3. System norms: H_2 and H_∞ spaces

In this section we discuss the normed spaces H_2 and H_∞. First, we introduce the $L_2(j\mathbb{R})$ space.

Definition 1.8 ($L_2(j\mathbb{R})$ space). $L_2(j\mathbb{R})$ is a Hilbert space of matrix-valued functions on $j\mathbb{R}$ and consists of all complex matrix functions F such that

$$\int_{-\infty}^{\infty} \mathrm{trace}[F^*(j\omega)F(j\omega)]d\omega < \infty.$$

The inner product is defined as

$$\langle F, G \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{trace}[F^*(j\omega)G(j\omega)]d\omega$$

for $F, G \in L_2(j\mathbb{R})$ of the same dimension.

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[5]We assume that the input, the output, and the state are initially at rest signals.

1. Introduction and Preliminaries

The H_2 space is defined in the following definition.

Definition 1.9 (H_2 space). H_2 is the space of matrix functions $F(s)$ that are analytic[6] in $\text{Re}(s) > 0$ and

$$\|F(s)\|_2^2 = \sup_{\sigma > 0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}[F^*(\sigma + j\omega)F(\sigma + j\omega)]d\omega\right\} < \infty.$$

Every element of H_2 may be identified with an element of $L_2(j\mathbb{R})$, which is denoted by the same symbol F. This means that H_2 may be seen as a subset of $L_2(j\mathbb{R})$. It may be shown that if $F(s) \in H_2$,

$$\|F(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}[F^*(j\omega)F(j\omega)]d\omega.$$

$H_2^\perp$ is the space of matrix-valued functions that are analytic in $\text{Re}(s) < 0$ and

$$\sup_{\sigma < 0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}[F^*(\sigma + j\omega)F(\sigma + j\omega)]d\omega\right\} < \infty.$$

Similar to the case of H_2, every element of $H_2^\perp$ may also be identified with an element of $L_2(j\mathbb{R})$. Thus $H_2^\perp$ may also be seen as a subspace of $L_2(j\mathbb{R})$. In fact, $H_2^\perp$ is the orthogonal complement of H_2 in $L_2(j\mathbb{R})$.

An important note is that the space $L_2(j\mathbb{R})$ can be related to the space $L_2(-\infty, \infty)$ in time domain. It can be shown that the two-sided Laplace transform (for $s = j\omega$) gives an isometric isomorphism between the $L_2(j\mathbb{R})$ in frequency domain and the $L_2(-\infty, \infty)$. The same is also true for H_2 and $L_2[0, \infty)$, and for $H_2^\perp$ and $L_2(-\infty, 0]$. Hence, the H_2-norm of a system F may also be computed from its impulse response matrix:

$$\|F\|_2^2 = \|F(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}[F^*(j\omega)F(j\omega)]d\omega$$

$$= \|F(t)\|_2^2 = \int_{-\infty}^{\infty}\text{trace}[F^*(t)F(t)]dt.$$

Furthermore, since the k-th column of the impulse response matrix is the response $y(t)$ to the input

$$u(t) = \begin{bmatrix} 0 \\ \vdots \\ u_k(t) = \delta(t) \\ \vdots \\ 0 \end{bmatrix},$$

---

[6] A complex valued function $f(s)$ defined on an open set $S \subseteq \mathbb{C}$ is said to be *analytic* in S if it is differentiable at every point in S.

$\|F\|_2$ may also be computed using

$$\|F\|_2^2 = \sum_{u=(0,\ldots,\delta(t),\ldots,0)} \int_{-\infty}^{\infty} y(t)^T y(t) dt.$$

Hence, the squared H_2-norm of a system F may also be computed by conducting p experiments, with p the dimension of the input u, as described in what follows. For the k-th experiment, the k-th element of u is set to the delta function $\delta(t)$ while the other elements are set to zero. The squared H_2-norm of the system F is then obtained by summing up the squared L_2-norm of the output y for all p experiments.

The H_2 norm also has a stochastic interpretation: it measures the power of the output when the system is driven by a unit intensity white noise input. To be precise, it may be shown that if the input u is wide sense stationary and[7]

$$\mathrm{E}[u(t)u(\tau)^T] = I\delta(t-\tau),$$

then

$$\|F\|_2^2 = \mathrm{E}\left[y(t)^T y(t)\right].$$

Next, we define the $L_\infty(j\mathbb{R})$ and H_∞ spaces.

Definition 1.10 ($L_\infty(j\mathbb{R})$ space). $L_\infty(j\mathbb{R})$ space is the Banach space of matrix-valued functions that are bounded on $j\mathbb{R}$, with norm[8]

$$\|F\|_\infty := \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)].$$

Definition 1.11 (H_∞ space). H_∞ is the space of matrix-valued functions that are analytic in the open right-half plane and

$$\|F\|_\infty := \sup_{\mathrm{Re}(s)>0} \bar{\sigma}[F(s)] < \infty.$$

Every element of H_∞ may be identified with an element of $L_\infty(j\mathbb{R})$, which is denoted by the same symbol F, so that H_∞ may be seen as a subset of $L_\infty(j\mathbb{R})$. It may be shown that if $F(s) \in H_\infty$,

$$\|F(s)\|_\infty = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)].$$

Furthermore, the H_∞ norm is equal to the induced 2-norm:

$$\|F(s)\|_\infty = \sup_{\|u(t)\|_2=1} \|y(t)\|_2,$$

given that $y = Fu$.

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[7]The operator E denotes the expectation.
[8]The operator $\bar{\sigma}$ denotes the maximum singular value.

1.3.4. Causality and properness

A system is called causal if the output does not depend on future input.

Definition 1.12 (Causality). A system F is said to be *causal* if the output up to time T depends only on the input up to time T, for every T. That is, F is causal if $P_T F P_T = P_T F$ for every T, in which P_T is an operator defined as

$$(P_T u)(t) = \left\{ \begin{array}{ll} u(t), & t \leq T \\ 0, & t > T. \end{array} \right.$$

In this thesis we use a definition of (strict) properness from [MZ00].

Definition 1.13 (Properness). A transfer matrix $F(s)$ is said to be *proper* if it is bounded in some right half plane in the sense that there is a $\rho \in \mathbb{R}$ such that

$$\sup_{\mathrm{Re}(s) > \rho} \bar{\sigma}[F(s)] < \infty.$$

Furthermore, it is said to be *strictly proper* if there is a $\rho \in \mathbb{R}$ such that

$$\lim_{|s| \to \infty \, \mathrm{Re}(s) > \rho} \bar{\sigma}[F(s)] = 0.$$

It turns out that properness is related to causality. A result from [Wei94] stipulates that a system has a causal implementation if its transfer matrix is proper in the sense of Definition 1.13.

1.3.5. Stability

Definition 1.14 (Stability). A system F with input u and output y is called *stable* if $y \in L_2[0, \infty)$ whenever $u \in L_2[0, \infty)$. It is called bistable if both F and F^{-1} are stable.

For linear time invariant systems, stability as defined above turns out to be equivalent to the transfer matrix being in H_∞ (see [Wei91]). Since the rational subspace of H_∞, denoted by RH_∞, contains proper rational transfer matrices with no poles in the closed right half plane, for proper rational transfer matrices, stability is equivalent to the absence of poles in the closed right half plane. In this thesis, the statements $'F(s)$ is stable' and $'F(s) \in H_\infty'$ are used interchangeably.

For state space systems, a well known result stipulates that a state space system described by (1.1,1.2) with (A, B) stabilizable and (A, C) detectable is stable in the sense of Definition 1.14 if and only if A is Hurwitz.
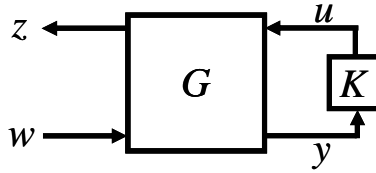
Figure 1.3.: The left homographic transformation.

1.3.6. Linear fractional transformation

A lower and upper linear fractional transformation (LFT) of two transfer matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and K of appropriate dimension are defined respectively as

$$F_\ell(P, K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$
$$F_u(P, K) := P_{22} + P_{21}K(I - P_{11}K)^{-1}P_{12}.$$

A lower LFT is said to be *well-posed* if $(I - P_{22}K)^{-1}$ exists and proper. Similarly an upper LFT is said to be well-posed if $(I - P_{11}K)^{-1}$ exists and proper.

The lower LFT (and upper LFT for that matter) may be represented as a block diagram interconnecting P and K, as shown in Figure 1.2 on page 3. In this block diagram, $F_\ell(P, K)$ is the transfer function that represents the mapping from w to z. As discussed earlier, the lower LFT is used in this thesis as the framework for control system analysis and design.

Another way to represent a feedback interconnection is via the chain scattering representation. In this framework, the left homographic transformation of two transfer matrices

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

and K of appropriate dimensions is defined as

$$C_\ell(G, K) := -\left(G_{11} - KG_{21}\right)^{-1}\left(G_{12} - KG_{22}\right),$$

provided that $(G_{11} - KG_{21})^{-1}$ exists. Figure 1.3 shows a block diagram representing the left homographic transformation. Here, we have that

$$\begin{bmatrix} z \\ w \end{bmatrix} = G \begin{bmatrix} u \\ y \end{bmatrix}, \quad u = Ky,$$

and the mapping from w to z is given by $C_\ell(G, K)$.

1.3.7. Algebraic Riccati equation

In this thesis, an algebraic Riccati equation is a matrix equation of the form

$$A^T X + XA + XRX + Q = 0,$$

where A, Q, and R are real $n \times n$ matrices with Q and R symmetric. A solution X of the Riccati equation is called *stabilizing* if $A + RX$ is Hurwitz. The following theorems states the conditions for which a (unique) stabilizing solution exists.

Theorem 1.15. *Suppose that (A, B) is stabilizable and and (C, A) has no unobservable modes on the imaginary axis. Then the Riccati equation*

$$A^T X + XA - XBB^T X + C^T C = 0$$

has a unique non-negative definite solution. In addition, the solution is stabilizing. Furthermore, the solution is positive definite if and only if (C, A) has no stable unobservable modes.

Proof. See Theorem 13.7 of [ZDG96]. □

Theorem 1.16. *Suppose D has full column rank and denote $R = D^T D > 0$. Then the Riccati equation*

$$(A - BR^{-1}D^T C)^T X + X(A - BR^{-1}D^T C)$$
$$- XBR^{-1}B^T X + C^T(I - DR^{-1}D^T)C = 0$$

has a unique nonnegative definite solution if and only if

- *(A, B) is stabilizable and*

- *$\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all ω.*

In addition, the solution is stabilizing. Furthermore, the solution is positive definite if and only if[9] $(D_\perp^T C, A - BR^{-1}D^T C)$ has no stable unobservable modes.

Proof. See Corollary 13.10 of [ZDG96]. □

1.3.8. Truncation and completion

Borrowing form [Mir03a], we define the *truncation* operator, which is denoted by $\tau_h(\cdot)$. The operation basically sets the value of the impulse response to zero after time equal to h time units. Given $F(s) = C(sI - A)^{-1}B$ we may express the truncation of F as:

$$\tau_h(F) := \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] - e^{-sh} \left[\begin{array}{c|c} A & B \\ \hline Ce^{Ah} & 0 \end{array} \right].$$

---

[9] $D_\perp$ is a matrix such that $\begin{bmatrix} D_\perp & DR^{-\frac{1}{2}} \end{bmatrix}$ is unitary and that $D_\perp D_\perp^T = I - DR^{-1}D$.
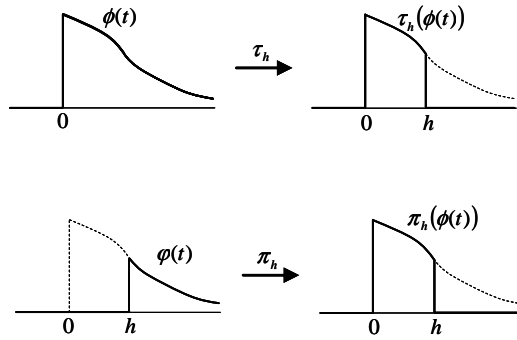
Figure 1.4.: Truncation (top) and completion (bottom) operations.

Also from [Mir03a], the *completion* operator, denoted by $\pi_h(\cdot)$, operates on delayed rational transfer function of the form $e^{-sh}F(s)$ and computes a FIR system such that the sum of the FIR system and $e^{-sh}F(s)$ is a rational transfer function. For $F(s) = C(sI - A)^{-1}B$, the completion operator is defined as

$$\pi_h(e^{-sh}F) := \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right] - e^{-sh}\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$ (1.5)

The truncation and completion operations are illustrated in Figure 1.4.

1. Introduction and Preliminaries

2

Youla-Kučera Parametrization

Internal stability is a basic requirement for a feedback system. It guarantees that nonzero initial conditions or errors in some location in the loop do not lead to unbounded signals anywhere else in the feedback loop. There are interesting questions that arise from the concept of internal stability:

- What are the conditions for the existence of a stabilizing controller?

- Is it unique? If not, is it possible to describe all stabilizing controllers?

For finite dimensional systems and systems with a single delay, the answers to these questions are completely understood (see e.g. [ZD98], [GL95], [MO79], [MR03]). This chapter provides the answers for systems with multiple i/o delays. The aim is to find all controllers that internally stabilize the standard control system with i/o delays shown in Figure 2.1.

The results in this chapter are based on the paper [MM03]. Following [MR03], the approach is to reduce the problem to an equivalent finite dimensional stabilization problem. In the equivalent problem, the plant is rational with no delay components. A mapping between the signals of the two closed loop systems is then formulated. Based on the mapping, the results from finite dimensional systems theory may be used to derive the parametrization.
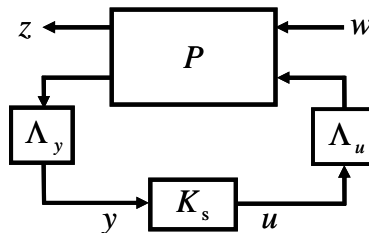
Figure 2.1.: Standard control system with i/o delays.

2.1. Literature review

In the mid 70s, Youla et. al. [YJB76] and Kučera [Kuč75] independently derived the parametrization of all stabilizing controller for finite dimensional systems. Based on a priori knowledge of one stabilizing controller, they described all stabilizing controllers parameterized by the set of all stable proper transfer functions. Although it has the reputation of being a somewhat under-utilized tool for practical control system design, the Youla-Kučera parametrization has been employed in addressing many control problems, including H_∞ and H_2 problems [And98].

The problem of stabilizing time-delay systems has been considered for many years. Smith [Smi57] managed to reduce the stabilization problem of stable SISO plants with a delay to a finite dimensional stabilization problem. The resulting controller later became known as the Smith predictor. Over the years, there have been many modifications to the Smith predictor. These include the observer-predictor form, which possess better disturbance rejection properties, and modifications to allow unstable plants (see [Pal96] for the references). The Smith predictor has also been extended to the case of MIMO plants with multiple delays. For a review of the Smith predictor and its modifications, the reader is advised to consult [Pal96] and the references therein. Another notable result is the paper [MO79], in which the problem of finite spectrum assignment for rational systems with a single delay is considered.

However, the papers mentioned in the preceding paragraph construct only one stabilizing controller for systems with a single delay and do not address the problem of parameterizing all stabilizing controllers. This problem is worked out in [MR03]. Using loop shifting arguments from [ZB97] and [CZ96], Mirkin and Raskin [MR03] came up with an elegant way to transform the problem of stabilizing systems with a single delay to a finite dimensional stabilization problem. A state-space parametrization of all stabilizing controllers may readily be obtained using standard finite dimensional systems theory.

The works [Ras00] and [MM03] independently extend the work in [MR03] to systems with multiple i/o delays.

2.2. Problem formulation

For an LFT configuration of Figure 2.1, the internal stability analysis is carried out by injecting a new external signal at each exposed interconnection point. The signal right after each injection point is then designated as a new output. The control system is said to be internally stable if all transfer functions from the inputs (including the new ones) to the outputs (including the new ones) are stable. The following definition clarifies the concept.
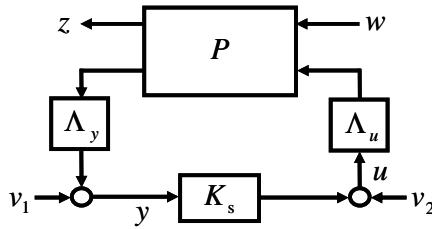
Figure 2.2.: Internal stability setup for an LFT with i/o delays.

Definition 2.1 (Internal stability). Consider the control system of Figure 2.1. The control system is said to be *internally stable* if the transfer matrix from the input signals (v_1, v_2, w) to the output signal (y, u, z) in the internal stability setup of Figure 2.2 is stable.

It is the aim of this chapter to derive a parametrization of all proper controllers that internally stabilize the control system of Figure 2.1. It is formally stated in what follows.

Problem 2.2 (Youla-Kučera parametrization). Consider the standard control system with i/o delays of Figure 2.1. Find a parametrization of all proper controllers that internally stabilize the control system of Figure 2.1, i.e. find all proper controllers that makes the transfer matrix from (v_1, v_2, w) to (y, u, z) in the internal stability setup of Figure 2.2 stable.

The plant $P(s)$ is assumed to have a stabilizable and detectable realization of the form

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & B_{P1} & B_{P2} \\ \hline C_{P1} & D_{P11} & D_{P12} \\ C_{P2} & D_{P21} & 0 \end{array} \right] \tag{2.1}$$

interconnected with a proper controller $K_s(s)$[1], and two multiple delay operators $\Lambda_y(s)$ and $\Lambda_u(s)$. The delay operators are of the form

$$\Lambda_y(s) = \mathrm{diag}(e^{-sh_{y1}}, e^{-sh_{y2}}, ..., e^{-sh_{ym}}) \tag{2.2}$$

$$\Lambda_u(s) = \mathrm{diag}(e^{-sh_{u1}}, e^{-sh_{u2}}, ..., e^{-sh_{up}}) \tag{2.3}$$

where m and p are dimension of y and u, respectively.

The D matrix of $P_{22}(s)$, denoted by D_{P22}, is assumed to be zero. If D_{P22} is nonzero, we may first find the controller for the case where it is zero and later recover the controller for the nonzero case by connecting $\Lambda_y(s)D_{P22}\Lambda_u(s)$ as a negative feedback to the controller. The details are

---

[1]The subscript s in K_s stands for 'standard'.

in what follows. First, the delay operators are absorbed into the plant P. The (2,2) block of the plant may be written as

$$\Lambda_y(s)P_{22}(s)\Lambda_u(s) = \Lambda_y(s)P_{22,\text{sp}}(s)\Lambda_u(s) + \Lambda_y(s)D_{P22}(s)\Lambda_u(s),$$

where $P_{22,\text{sp}}(s)$ is the strictly proper part of $P_{22}(s)$. The non-strictly-proper part, namely $\Lambda_y(s)D_{P22}(s)\Lambda_u(s)$, may be absorbed to the controller $K_\text{s}(s)$ as a positive feedback interconnection. We may then find the controller for the new plant, which has a strictly proper (2,2) block. The original controller may be recovered by interconnecting it with $\Lambda_y(s)D_{P22}(s)\Lambda_u(s)$, this time as a negative feedback interconnection[2].

2.3. Equivalent finite dimensional stabilization problem

Following the procedure in [MR03], an algorithm for transforming the stabilization setup of Figure 2.2 to an equivalent finite dimensional stabilization setup is developed.

Let us first introduce a decomposition of a rational transfer matrix that is pre and post multiplied by delay operators to a sum of another rational transfer matrix and a stable but non-rational transfer matrix. The latter may be chosen to have finite impulse response. The decomposition corresponding to the plant (2.1) and the delay operators (2.2,2.3) is given in the following definition.

Definition 2.3 (Rational-stable decomposition). The transfer matrices $\Phi_{12}(s)$, $\Phi_{21}(s)$, $\Phi_{22}(s)$, $\tilde{P}_{12}(s)$, $\tilde{P}_{21}(s)$, and $\tilde{P}_{22}(s)$ are defined such that the following conditions are satisfied:

- The equations

$$P_{12}(s)\Lambda_u(s) = \tilde{P}_{12}(s) - \Phi_{12}(s) \tag{2.4}$$

$$\Lambda_y(s)P_{21}(s) = \tilde{P}_{21}(s) - \Phi_{21}(s) \tag{2.5}$$

$$\Lambda_y(s)P_{22}(s)\Lambda_u(s) = \tilde{P}_{22}(s) - \Phi_{22}(s) \tag{2.6}$$

are satisfied,

- $\tilde{P}_{12}(s)$, $\tilde{P}_{21}(s)$ and $\tilde{P}_{22}(s)$ are rational,

- $\Phi_{12}(s)$, $\Phi_{21}(s)$, and $\Phi_{22}(s)$ are stable, and $\Phi_{22}(s)$ is strictly proper.

---

[2] Note that if some components of $\Lambda_y(s)D_{P22}(s)\Lambda_u(s)$ are non-delayed, then for certain Youla parameter Q (see Theorem 2.8), the resulting controller may be non-proper.
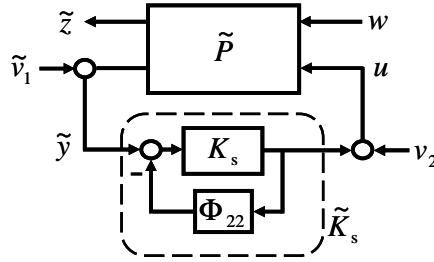
Figure 2.3.: The equivalent finite dimensional stabilization setup.

A way to construct the transfer matrices according to Definition 2.3 will be elaborated later in this section.

The next two lemmas show that the stabilization setup of Figure 2.2 may be transformed to an equivalent finite dimensional stabilization setup of Figure 2.3 where

$$\tilde{P}(s) = \left[\begin{array}{cc} \tilde{P}_{11}(s) & \tilde{P}_{12}(s) \\ \tilde{P}_{21}(s) & \tilde{P}_{22}(s) \end{array} \right] \tag{2.7}$$

and

$$\tilde{K}_{\mathsf{s}}(s) = (I + K_{\mathsf{s}}(s)\Phi_{22}(s))^{-1}K_{\mathsf{s}}(s). \tag{2.8}$$

Here, the two setups are equivalent in the sense that there is a one to one relation between the signals in the two setups and between $K_{\mathsf{s}}(s)$ and $\tilde{K}_{\mathsf{s}}(s)$.

The first lemma shows that the controller $\tilde{K}_{\mathsf{s}}(s)$ in Figure 2.3 is well-defined and proper, while the second lemma demonstrates that the configuration in Figure 2.2 and Figure 2.3 have an equivalent internal stability property, in the sense that the internal stability of one configuration is equivalent to the internal stability of the other.

Lemma 2.4 (Proper bijection $K_{\mathsf{s}} \sim \tilde{K}_{\mathsf{s}}$). *Let $\Phi_{22}(s)$ be defined as in Definition 2.3. Define*

$$\tilde{K}_s(s) = (I + K_s(s)\Phi_{22}(s))^{-1}K_s(s). \tag{2.9}$$

Then $K_s(s)$ may be inferred from $\tilde{K}_s(s)$ using the equation

$$K_s(s) = (I - \tilde{K}_s(s)\Phi_{22}(s))^{-1}\tilde{K}_s(s). \tag{2.10}$$

Furthermore, $\tilde{K}_s(s)$ is well-defined and proper if and only if $K_s(s)$ is proper.

Proof. Using simple algebraic manipulations, (2.10) may be obtained. To prove that $K(s)$ is proper if and only if $\tilde{K}_{\mathsf{s}}(s)$ is proper, first notice that $\Phi_{22}(s)$ is strictly proper. Suppose $K_{\mathsf{s}}(s)$ is proper, then there exists a $\rho \in \mathbb{R}$ such that

$$\lim_{s\to\infty,\mathrm{Re}(s)>\rho}(I + K_{\mathsf{s}}(s)\Phi_{22}(s))^{-1} = I, \tag{2.11}$$
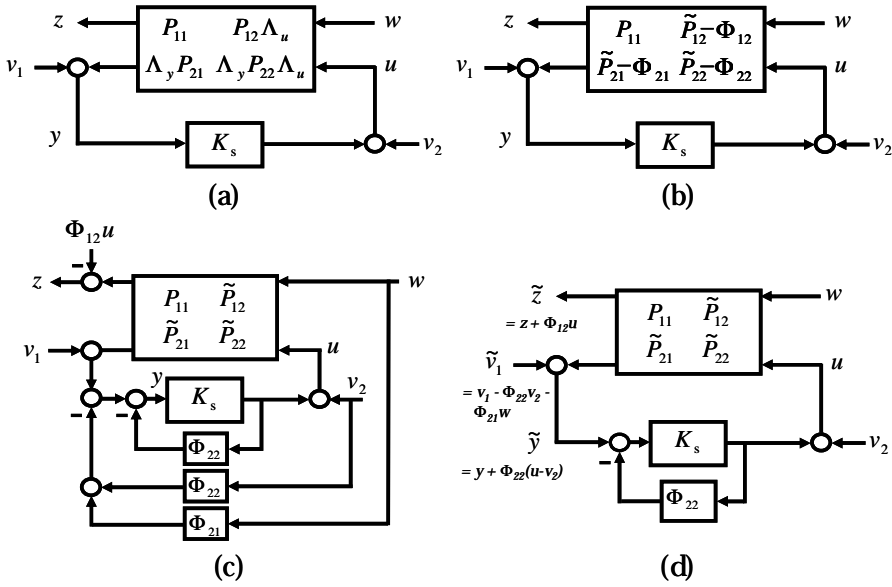
Figure 2.4.: The proof of Lemma 2.5

implying that $\tilde{K}_{\mathrm{s}}(s) = (I + K_{\mathrm{s}}(s)\Phi_{22}(s))^{-1}K_{\mathrm{s}}(s)$ is proper. The converse may be proved using similar arguments. □

Lemma 2.4 shows that the relation between $K_{\mathrm{s}}(s)$ and $\tilde{K}_{\mathrm{s}}(s)$ is a bijection.

Lemma 2.5 (Internal stability equivalence). *Define the plant with the delay operators absorbed:*

$$\hat{P}(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s)\Lambda_u(s) \\ \Lambda_y(s)P_{21}(s) & \Lambda_y(s)P_{22}(s)\Lambda_u(s) \end{bmatrix}, \tag{2.12}$$

where $P(s)$ is given by (2.1), while $\Lambda_y(s)$ and $\Lambda_u(s)$ are given by (2.2) and (2.3), respectively. Then the controller $K_s(s)$ internally stabilizes $\hat{P}(s)$, i.e. the control system of Figure 2.1 is internally stable, if and only if $\tilde{K}_s(s)$ defined in (2.9) internally stabilizes the rational plant $\tilde{P}(s)$ as defined in (2.7).

Proof. Consider the setup of Figure 2.2. By absorbing the delay matrices to the plant, we obtain Figure 2.4(a). Now, observe Figure 2.4(a)-(d). In this figure, it is shown that the setup in Figure 2.2, through block diagram manipulations similar to the technique used in [ZB97],[MR03] and the use of the equations (2.4,2.5,2.6), may be transformed to the setup of Figure 2.3

with

$$\tilde{v}_1 = v_1 - \Phi_{22}(s)v_2 - \Phi_{21}(s)w, \tag{2.13}$$

$$\tilde{z} = z + \Phi_{12}(s)u, \tag{2.14}$$

$$\tilde{y} = y + \Phi_{22}(s)(u - v_2). \tag{2.15}$$

The equations (2.13,2.14,2.15) suggest that the output signals $(\tilde{y}, u, \tilde{z})$ of Figure 2.3 may be expressed in terms of the output signals (y, u, z) and the input signals (v_1, v_2, w) of Figure 2.2 and vice versa:

$$\begin{bmatrix} \tilde{y} \\ u \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} I & \Phi_{22}(s) & 0 \\ 0 & I & 0 \\ 0 & \Phi_{12}(s) & I \end{bmatrix} \begin{bmatrix} y \\ u \\ z \end{bmatrix} + \begin{bmatrix} 0 & -\Phi_{22}(s) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ w \end{bmatrix} \tag{2.16}$$

$$\begin{bmatrix} y \\ u \\ z \end{bmatrix} = \begin{bmatrix} I & -\Phi_{22}(s) & 0 \\ 0 & I & 0 \\ 0 & -\Phi_{12}(s) & I \end{bmatrix} \begin{bmatrix} \tilde{y} \\ u \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} 0 & \Phi_{22}(s) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ v_2 \\ w \end{bmatrix}. \tag{2.17}$$

Notice that all transfer matrices in the equations (2.16,2.17) are stable. Moreover, the equation (2.13) also suggests a bistable mapping between the input signals of the two stabilization setups:

$$\begin{bmatrix} \tilde{v}_1 \\ v_2 \\ w \end{bmatrix} = \begin{bmatrix} I & -\Phi_{22}(s) & -\Phi_{21}(s) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ w \end{bmatrix} \tag{2.18}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ w \end{bmatrix} = \begin{bmatrix} I & \Phi_{22}(s) & \Phi_{21}(s) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ v_2 \\ w \end{bmatrix}. \tag{2.19}$$

The bistable mappings allow us to prove the lemma. Let us denote $T(s)$ as the transfer matrix from (v_1, v_2, w) to (y, u, z) in Figure 2.2, i.e.

$$\begin{bmatrix} y \\ u \\ z \end{bmatrix} = T(s) \begin{bmatrix} v_1 \\ v_2 \\ w \end{bmatrix}. \tag{2.20}$$

Suppose $K_{\mathrm{s}}(s)$ stabilizes $\hat{P}(s)$, then $T(s)$ is stable. Substituting (2.17) and (2.19) to (2.20) we obtain

$$\begin{bmatrix} \tilde{y} \\ u \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} I & \Phi_{22}(s) & 0 \\ 0 & I & 0 \\ 0 & \Phi_{12}(s) & I \end{bmatrix} \left(T(s) \begin{bmatrix} I & \Phi_{22}(s) & \Phi_{21}(s) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} 0 & \Phi_{22}(s) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \tilde{v}_1 \\ v_2 \\ w \end{bmatrix}, \tag{2.21}$$

showing the transfer function from $(\tilde{v}_1, v_2, w)$ to $(\tilde{y}, u, \tilde{z})$ in Figure 2.3 is stable. It follows that $\tilde{K}_{\mathrm{s}}(s)$ stabilizes $\tilde{P}(s)$. The converse may be proved in a similar manner. □

Lemma 2.4 and Lemma 2.5 established the desired result of converting the infinite dimensional stabilization problem into a finite dimensional one. What remains is to construct the transfer matrices $\Phi_{12}(s)$, $\Phi_{21}(s)$, $\Phi_{22}(s)$, $\tilde{P}_{12}(s)$, $\tilde{P}_{21}(s)$, and $\tilde{P}_{22}(s)$ according to Definition 2.3. First recall the rational plant $P(s)$ and the delay matrices $\Lambda_u(s)$ and $\Lambda_y(s)$:

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & B_{P1} & B_{P2} \\ \hline C_{P1} & D_{P11} & D_{P12} \\ C_{P2} & D_{P21} & 0 \end{array}\right], \qquad (2.22)$$

$$\Lambda_y(s) = \mathrm{diag}(e^{-sh_{y1}}, e^{-sh_{y2}}, ..., e^{-sh_{ym}}), \qquad (2.23)$$

$$\Lambda_u(s) = \mathrm{diag}(e^{-sh_{u1}}, e^{-sh_{u2}}, ..., e^{-sh_{up}}). \qquad (2.24)$$

The following proposition provides a way to construct the desired transfer matrices.

Proposition 2.6 (Rational-FIR decomposition). *Let $\tilde{P}_{12}(s)$, $\tilde{P}_{21}(s)$, and $\tilde{P}_{22}(s)$ be given by*

$$\tilde{P}_{12}(s) = \left[\begin{array}{c|c} A_P & \tilde{B}_{P2} \\ \hline C_{P1} & D_{P12} \end{array}\right] \qquad (2.25)$$

$$\begin{bmatrix} \tilde{P}_{21}(s) & \tilde{P}_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & B_{P1} & \tilde{B}_{P2} \\ \hline \tilde{C}_{P2} & D_{P21} & 0 \end{array}\right] \qquad (2.26)$$

where

$$\tilde{C}_{P2} = \begin{bmatrix} C_{P2,1}e^{-A_P h_{y1}} \\ C_{P2,2}e^{-A_P h_{y2}} \\ \vdots \\ C_{P2,m}e^{-A_P h_{ym}} \end{bmatrix} \qquad (2.27)$$

$$\tilde{B}_{P2} = \begin{bmatrix} e^{-A_P h_{u1}}B_{P2,1} & e^{-A_P h_{u2}}B_{P2,2} & \dots & e^{-A_P h_{up}}B_{P2,p} \end{bmatrix}, \qquad (2.28)$$

with $B_{P2,i}$ and $C_{P2,i}$ are the i-th column of B_{P2} and i-th row of C_{P2}, respectively. Let $P_{12}(s)$, $P_{21}(s)$, and $P_{22}(s)$ be defined by (2.22). *Suppose that the transfer functions $\Phi_{12}(s)$, $\Phi_{21}(s)$, and $\Phi_{22}(s)$ are defined by* (2.4), (2.5), *and* (2.6), *respectively. Then the $\Phi_{12}(s)$, $\Phi_{21}(s)$, and $\Phi_{22}(s)$ are stable, and $\Phi_{22}(s)$ is strictly proper.*

Proof. Here, only the prove for the case of $\tilde{P}_{22}(s)$ is provided. The cases corresponding to $\tilde{P}_{12}(s)$ and $\tilde{P}_{21}(s)$ may be proved in a similar manner. First note that the impulse response of $\tilde{P}_{22}(s)$ is given by

$$\tilde{P}_{22}(t) = \tilde{C}_{P2}e^{A_P t}\tilde{B}_{P2}\mathbb{1}(t)$$
$$= \begin{bmatrix} C_{P2,1}e^{A_P(t-h_{y1}-h_{u1})}B_{P2,1} & \dots & C_{P2,1}e^{A_P(t-h_{y1}-h_{up})}B_{P2,p} \\ \vdots & \ddots & \vdots \\ C_{P2,m}e^{A_P(t-h_{ym}-h_{u1})}B_{P2,1} & \dots & C_{P2,m}e^{A_P(t-h_{ym}-h_{up})}B_{P2,p} \end{bmatrix} \mathbb{1}(t),$$
$$(2.29)$$

while the impulse response of $P_{22}(s)$ is given by

$$P_{22}(t) = \begin{bmatrix} C_{P2,1}e^{A_P t}B_{P2,1} & \dots & C_{P2,1}e^{A_P t}B_{P2,p} \\ \vdots & \ddots & \vdots \\ C_{P2,m}e^{A_P t}B_{P2,1} & \dots & C_{P2,m}e^{A_P t}B_{P2,p} \end{bmatrix} \mathbb{1}(t). \qquad (2.30)$$

The impulse response of $\Lambda_y(s)P_{22}(s)\Lambda_u(s)$ is given by $\Lambda_y(t) * P_{22}(t) * \Lambda_u(t)$, where

$$\Lambda_y(t) = \mathrm{diag}(\delta(t - h_{y1}), \dots, \delta(t - h_{ym})) \qquad (2.31)$$

$$\Lambda_u(t) = \mathrm{diag}(\delta(t - h_{u1}), \dots, \delta(t - h_{up})). \qquad (2.32)$$

It follows that the (i, j)-th component of the impulse response of the transfer function $\Lambda_y(s)P_{22}(s)\Lambda_u(s)$ is given by

$$\left(\Lambda_y(t) * P_{22}(t) * \Lambda_u(t)\right)_{i,j} = C_{P2,i}e^{A_P(t - h_{yi} - h_{uj})}B_{P2,j}\mathbb{1}(t - h_{yi} - h_{uj}). \qquad (2.33)$$

By (2.6), $\Phi_{22}(s) = \tilde{P}_{22}(s) - \Lambda_y(s)P_{22}(s)\Lambda_u(s)$ and its impulse response is given by

$$\Phi_{22}(t) = \tilde{P}_{22}(t) - \Lambda_y(t) * P_{22}(t) * \Lambda_u(t). \qquad (2.34)$$

The (i, j)-th component of $\Phi_{22}(t)$ is then given by

$$\left(\Phi_{22}(t)\right)_{i,j} = C_{P2,i}e^{A_P(t - h_{yi} - h_{uj})}B_{P2,j}\left(\mathbb{1}(t) - \mathbb{1}(t - h_{yi} - h_{uj})\right). \qquad (2.35)$$

It is evident that $\Phi_{22}(t)$ is a finite impulse response matrix with its (i, j)-th component having support on $[0, h_{yi} + h_{uj}]$. Hence, $\Phi_{22}(s) \in H_\infty$. The fact that $\Phi_{22}(s)$ is strictly proper follows from the fact that both $P_{22}(s)$ and $\tilde{P}_{22}(s)$ are strictly proper. $\qquad \square$

Proposition 2.6 provides a state-space realization of $\tilde{P}(s)$:

$$\tilde{P}(s) = \begin{bmatrix} P_{11}(s) & \tilde{P}_{12}(s) \\ \tilde{P}_{21}(s) & \tilde{P}_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & B_{P1} & \tilde{B}_{P2} \\ \hline C_{P1} & D_{P11} & D_{P12} \\ \tilde{C}_{P2} & D_{P21} & 0 \end{array} \right]. \qquad (2.36)$$

Using Lemma 2.5 and the realization of $\tilde{P}(s)$ given by (2.36), the condition for the existence of a stabilizing controller for the standard control system of Figure 2.1 may be obtained.

Lemma 2.7 (Stabilizability condition[3]). *There exists a proper controller $K_s(s)$ such that the control system of Figure 2.1, where $P(s)$, $\Lambda_y(s)$, and $\Lambda_u(s)$ are given by (2.1,2.2,2.3), is internally stable if and only if the pairs (A_P, B_{P2}) and (C_{P2}, A_P) are stabilizable and detectable, respectively.*

---

[3]For a plant interconnected with a controller in an LFT configuration, stabilizability means that there exists a controller such that the LFT is internally stable.

Proof. It follows from Lemma 2.5 that there exist a proper controller that internally stabilizes the control system of Figure 2.1 if and only if the rational plant $\tilde{P}(s)$ given by (2.36) is stabilizable by a proper controller. Standard theory (see e.g. [GL95]) says that the plant $\tilde{P}(s)$ is stabilizable if and only if the pairs $(A_P, \tilde{B}_{P2})$ and $(\tilde{C}_{P2}, A_P)$ are stabilizable and detectable, respectively.

The latter condition is equivalent to the pairs (A_P, B_{P2}) and (C_{P2}, A_P) being stabilizable and detectable. To see this, consider the system $\dot{x}(t) = A_P x(t), x(0) = x_0, y(t) = C_{P2}x(t)$. One definition of detectability is the property that if the output is identically zero, i.e. $y(t) = C_{P2}x(t) \equiv 0$, then $x(t) \to 0$ (see e.g. [Son98]). Now consider the same system that shares the same state but with a slightly different output: $\dot{x}(t) = A_P x(t), x(0) = x_0, \tilde{y}(t) = \tilde{C}_{P2}x(t)$. Suppose that the output $\tilde{y}(t) \equiv 0$. We have that

$$\tilde{y}(t) = \tilde{C}_{P2}x(t) = \tilde{C}_{P2}e^{A_P t}x_0 = \begin{bmatrix} y_1(t - h_{y1}) \\ \vdots \\ y_m(t - h_{ym}) \end{bmatrix} \equiv 0, \qquad (2.37)$$

where $y_i(t)$ is the i-th component of $y(t)$. This shows that $y(t) = C_{P2}x(t) \equiv 0$ if and only if $\tilde{y}(t) = \tilde{C}_{P2}x(t) \equiv 0$. Hence, the detectability of the pair (C_{P2}, A_P) is equivalent to the detectability of the pair $(\tilde{C}_{P2}, A_P)$. Using the duality of detectability and stabilizability, we arrive at the same conclusion for the equivalence of the stabilizability of the pairs (A_P, B_{P2}) and $(A_P, \tilde{B}_{P2})$. □

2.4. Youla-Kučera parametrization

To obtain the parametrization of all stabilizing controller $K_s(s)$ for the LFT in figure 2.1, Lemma 2.4 and 2.5 suggest to first compute the stabilizing controller $\tilde{K}_s(s)$ for the rational plant $\tilde{P}(s)$, and then use (2.10), i.e. adding a positive feedback with feedback gain matrix $\Phi_{22}(s)$, to obtain $K_s(s)$. To verify the existence of a stabilizing controller, Lemma 2.7 may be used.

The above discussion results in the following theorem.

Theorem 2.8 (Youla-Kučera parametrization). *Consider the control system of Figure 2.1 where the plant $P(s)$ has a stabilizable and detectable realization of (2.1), while the delay operators $\Lambda_y(s)$ and $\Lambda_u(s)$ are given by (2.2) and (2.3), respectively. There exists a proper controller $K_s(s)$ such that the control system of Figure 2.1 is internally stable if and only if the pairs (A_P, B_{P2}) and (C_{P2}, A_P) are stabilizable and detectable, respectively. Furthermore, define F_Y and L_Y such that $(A_P + \tilde{B}_{P2}F_Y)$ and $(A_P + L_Y\tilde{C}_{P2})$ are Hurwitz with*

$$\tilde{C}_{P2} = \begin{bmatrix} e^{-A_P^T h_{y1}}C_{P2,1}^T & \dots & e^{-A_P^T h_{ym}}C_{P2,m}^T \end{bmatrix}^T \qquad (2.38)$$
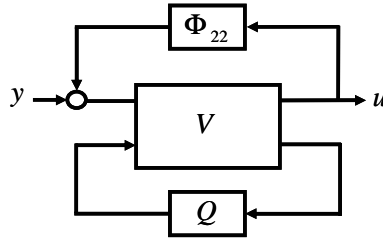
Figure 2.5.: Youla-Kučera parametrization for control systems with multiple i/o delays.

$$\tilde{B}_{P2} = \begin{bmatrix} e^{-A_P h_{u1}} B_{P2,1} & \dots & e^{-A_P h_{up}} B_{P2,p} \end{bmatrix} \tag{2.39}$$

where $B_{P2,i}$ and $C_{P2,i}$ are the i-th column of B_{P2} and i-th row of C_{P2}, respectively.

Then all proper controllers that internally stabilize the control system in Figure 2.1 are parameterized as described in Figure 2.5, where

$$V(s) = \left[\begin{array}{c|cc} A_P + \tilde{B}_{P2}F_Y + L_Y\tilde{C}_{P2} & -L_Y & \tilde{B}_{P2} \\ \hline F_Y & 0 & I \\ -\tilde{C}_{P2} & I & 0 \end{array} \right] \tag{2.40}$$

$$\Phi_{22}(s) = \left[\begin{array}{c|c} A_P & \tilde{B}_{P2} \\ \hline \tilde{C}_{P2} & 0 \end{array} \right] - \Lambda_y(s) \left[\begin{array}{c|c} A_P & B_{P2} \\ \hline C_{P2} & 0 \end{array} \right] \Lambda_u(s) \tag{2.41}$$

and with $Q(s) \in H_\infty$ but otherwise arbitrary.

Proof. By Lemma 2.7, there exists a stabilizing controller for the control system in Figure 2.1 if and only if the pairs (A_P, B_{P2}) and (C_{P2}, A_P) are stabilizable and detectable, respectively.

Applying Lemma A.4.5 of [GL95] to the realization of $\tilde{P}(s)$ defined in (2.36), we obtain the parametrization of all stabilizing controller $\tilde{K}_s(s) = F_\ell(V(s), Q(s))$, where $V(s)$ is given by (2.40) and $Q(s)$ is any stable transfer function.

By Lemma 2.5, all stabilizing controller $K_s(s)$ for the control system in Figure 2.1 is obtained through equation (2.10) , i.e. by adding a positive feedback with feedback gain matrix $\Phi_{22}(s)$ to $\tilde{K}_s(s)$. This results in the controller in Figure 2.5. Finally, Lemma 2.4 ensures that $K_s(s)$ is proper. □

Remark 2.9 (The case with nonzero D_{P22}). It is mentioned earlier that if $D_{P22} \neq 0$, then for certain Youla parameter Q the resulting controller may be non-proper. This may be avoided for example by limiting Q to strictly proper transfer functions, which will ensure that the resulting controller is well-defined and proper.
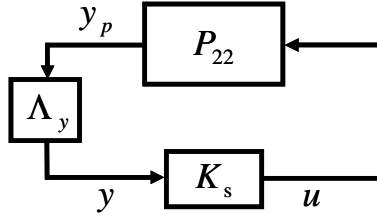
Figure 2.6.: Stabilization of $\Lambda_y(s)P_{22}(s)$.

2.5. An observer plus state feedback interpretation of the controller

In this section, an insight into how the controller works is presented. We show that in the absence of the external input w and the input delay matrix $\Lambda_u(s)$, the central controller estimates the state of the undelayed plant.

Assuming that a stabilizing controller $K_s(s)$ exists, it is known that a controller internally stabilizes a rational plant $P(s)$ if and only if it stabilizes the plant $(2, 2)$ part. Using Lemma 2.5, it may be shown that this is also true even if delays are present in the i/o channels. This configuration with $\Lambda_u(s) = I$ is depicted in Figure 2.6. Since it is a lot simpler than the original configuration with the full plant, it is easier to analyze how the controller works in this configuration. If we apply the parametrization of Theorem 2.8 to the delayed plant $\Lambda_y(s)P_{22}(s)$ and then set the parameter $Q(s)$ to zero, we obtain the central controller. It has the form a feedback interconnection of a rational transfer function and $\Phi_{22}(s)$. The rational part is given by

$$V_c(s) = \left[\begin{array}{c|c} A_P + B_{P2}F_Y + L_Y\tilde{C}_{P2} & -L_Y \\ \hline F_Y & 0 \end{array}\right]. \tag{2.42}$$

It may be shown that $V_c(s)$ may be written in an observer-based form consisting an observer $O(s)$ and a stabilizing feedback $-F_Y$. The observer is of the form

$$O(s) = \left[\begin{array}{c|cc} A_P + L_Y\tilde{C}_{P2} & -L_Y & B_{P2} \\ \hline I & 0 & 0 \end{array}\right]. \tag{2.43}$$

Figure 2.7 shows the resulting closed-loop system. If we denote the state of $P_{22}(s)$ by x, then the undelayed measurements are given by $y_p = C_{P2}x$. Keeping in mind that $\Lambda_y(s)P_{22}(s) + \Phi_{22}(s) = \tilde{P}_{22}(s)$, it is evident that the dotted part of Figure 2.7 is actually $\tilde{P}_{22}(s)$, driven by the same input u that drives $P_{22}(s)$. Since $P_{22}(s)$ and $\tilde{P}_{22}(s)$ share the same A and B matrices, they share the same state variable x. In fact, the measurements that are fed
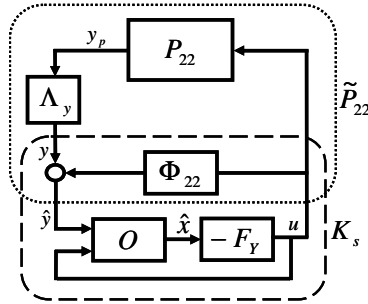
Figure 2.7.: The central stabilizing controller for $\Lambda_y(s)P_{22}(s)$ in an observer-based form.

to the observer $O(s)$ are given by $y = \tilde{C}_{P2}x$. The dynamics of the error between x and the output of the observer $\hat{x}$ are given by the familiar equation

$$\dot{x} - \dot{\hat{x}} = (A_P + L_Y \tilde{C}_{P2})(x - \hat{x}). \tag{2.44}$$

Since $(A_P + L_Y \tilde{C}_{P2})$ is Hurwitz, the error goes exponentially to zero as time goes to infinity. Hence, $\hat{x}$ is an estimate of the state of the undelayed plant $P_{22}(s)$.

From the above discussion, we may interpret the rationale of the controller as follows. Since only the delayed version of the output $y_p = C_{P2}x$ is available, the controller compensates the delayed measurements using the FIR block $\Phi_{22}(s)$ to obtain $\hat{y} = \tilde{C}_{P2}x$. Using the fact that $(\tilde{C}_{P2}, A_P)$ is detectable, the controller estimates the state x and uses it to stabilize the plant.

2.6. Concluding remarks

In this chapter, the result of converting the stabilization problem of systems with multiple i/o delays into a finite dimensional problem is established. This result allows the parametrization of all stabilizing controllers for systems with i/o delays, resulting in the Youla-Kučera parametrization. Later, it will be shown that this result is also needed to solve the H_2-optimal control problem, particularly for the frequency domain approach.

2. Youla-Kučera Parametrization

3

The Standard H_2 Problem: Time Domain Approach

In the previous chapter, the important question of stabilizing control systems with delays is addressed. However, the discussion does not touch the issue of the performance of the control system. Although the parametrization gives a lot of room for design through the free parameter Q, it is not clear how to choose Q such that a certain degree of performance is achieved. This chapter provides the theory to synthesize a controller that optimizes the control system performance in the H_2-sense.

Here, the standard LFT framework of Figure 2.1 (page 17) is used. The aim is to obtain a controller that minimizes the H_2-norm of the transfer function from the exogenous input w to the controlled output z. However, we limit our attention to the cases where the delays occur on only one side of the controller, i.e. either in the measurement output or in the control input. The input delay case and the output delay case are shown in Figure 3.1(a) and Figure 3.1(b), respectively.

The approach is to convert the standard problem to what is called the regulator problem, shown in Figure 3.2. The latter is then reformulated as a linear quadratic regulator (LQR) problem with multiple input delays. It is shown that the LQR problem with delays may be reduced to a number of standard LQR problems. It is done by splitting the optimization time interval into time regions compatible with the delays and then apply dynamic programming ideas. Within each time region, the optimization becomes essentially delay-free and may be solved using standard LQR methods.

This chapter is based on the papers [MM04] and [MM05c].

3.1. Literature review

Since the Smith predictor [Smi57], a lot of results have been put forward to solve control problems concerning systems with delays. In particular, there
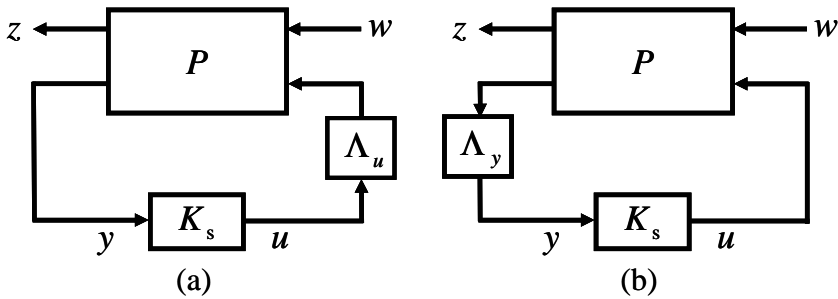
Figure 3.1.: The standard problem, (a) input delay case, (b) output delay case.

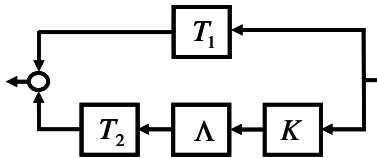

Figure 3.2.: The regulator problem.

have been numerous attempts to control systems with delays optimally in some sense.

In the area of H_∞ control, the books [FÖT96] and [vK93] treated a general class of infinite dimensional H_∞ control problems that include systems with delays. Later, methods that were specifically tailored for systems with i/o delays were developed. The single delay case was treated in [MZ00], [Mir03a], while the solution of multiple i/o delays case may be found in [MM05a], [KI03a]. In particular, the paper [Mir03a] treated the delay as an additional constraint imposed on the controller. This point of view results in the transformation of the standard problem to a simpler one-block problem. This technique is also used in this chapter.

Similar to the H_∞ case, in the area of H_2 (LQG) control, general infinite dimensional theory may also be applied to systems with delays. Chapter 6 of the book by Curtain and Zwart [CZ95] provides a detailed overview and references of the LQ theory for infinite dimensional systems. Along the infinite dimensional theory line, Delfour, McCalla and Mitter ([DM72a], [DM72b], [DMM75]) treated the LQ control of retarded differential equations. Extensions to general delay equations with delayed inputs and outputs may be found in the work of Pritchard and Salamon [PS85], [PS87] and Delfour [Del86]. Earlier, Kleinman [Kle69] came up with a solution of the LQG problem tailored for systems with a single i/o delay. Recently, it was shown by Mirkin and Raskin in [MR03] that the solution for this prob-

lem may be obtained by converting the infinite dimensional problem to an equivalent finite dimensional optimization using loop shifting techniques originated from [CZ96], [ZB97]. The multiple delays case was first considered in the paper [SR72], where the LQR problem with multiple input and state delays was first solved.

This chapter provides a time domain solution to H_2 output feedback optimal control for systems with multiple i/o delays. However, the solution only applies to cases where the delays are present only on one side of the controller. Here, the problem is solved by converting the standard problem to the regulator problem, which is then reformulated as an LQR problem with multiple input delays. The LQR problem is solved by splitting the optimization interval into time regions compatible with the delays. Note that the splitting technique has been previously applied in [Tad97a] for the single delay case in the context of robust control in the gap, in [Tad00] in the context of the standard H_∞ problem, and in [Tad97b] in solving a one-block problem. Apart from the one in [SR72], another solution of the LQR problem may be found in the paper by Kojima and Ishijima [KI03a], in which the problem is treated as the the limiting problem of an H_∞ problem. Compared to the derivation and formulas of the solution in [SR72], the derivation and formulas of the solution in this chapter appear simpler. However, the solution in [SR72] is more general since it covers the case with delays in the state.

However, these results do not cover the general case where delays are present both in the measurement output and in the control input. The general case is solved in [MMK05] (Chapter 4 of this thesis), where it is solved using frequency domain techniques.

3.2. The standard H_2 problem with one sided delays

We consider standard control systems in which time delays are present in either the control input or the measurement output. Such control systems are depicted in Figure 3.1(a) for the input delay case, and in Figure 3.1(b) for the output delay case. Here the plant $P(s)$ is a rational transfer matrix which is assumed of having the realization

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & B_{P1} & B_{P2} \\ \hline C_{P1} & 0 & D_{P12} \\ C_{P2} & D_{P21} & 0 \end{array} \right] \qquad (3.1)$$

connected with a proper controller $K_s(s)$, and a multiple delay operator. The output and input multiple delay operators are of the form

$$\Lambda_y(s) = \mathrm{diag}(e^{-sh_{y1}}, e^{-sh_{y2}}, ..., e^{-sh_{yp}}), \qquad (3.2)$$

$$\Lambda_u(s) = \mathrm{diag}(e^{-sh_{u1}}, e^{-sh_{u2}}, ..., e^{-sh_{um}}). \qquad (3.3)$$

where p and m are the dimension of y and u, respectively. We assume that $P(s)$ satisfies the following standard assumptions:

A1 (C_{P2}, A_P, B_{P2}) is detectable and stabilizable;

A2 $R_1 = D_{P12}^T D_{P12} > 0$ and $R_2 = D_{P21} D_{P21}^T > 0$,

A3 $\begin{bmatrix} A_P - j\omega I & B_{P2} \\ C_{P1} & D_{P12} \end{bmatrix}$ and $\begin{bmatrix} A_P - j\omega I & B_{P1} \\ C_{P2} & D_{P21} \end{bmatrix}$ have full column rank and full row rank, respectively $\forall \omega \in \mathbb{R}$.

The first assumption ensures that the plant P can be stabilized by output feedback, while the three assumptions together guarantee that the the Riccati equations associated with the H_2 problem, which is later defined, have a stabilizing solution. We also use the standard assumption of the upper left block of the D-matrix of P, denoted by D_{P11}, being zero. Without loss of generality, the lower right block, denoted by D_{P22}, is also assumed to be zero. If D_{P22} is nonzero, we may first find the controller for the case where it is zero and later recover the controller for the nonzero case by connecting $\Lambda_y(s)D_{P22}\Lambda_u(s)$ as a negative feedback to the controller. The details may be found in Section 2.2.

Problem 3.1 (Standard H_2 problem with one sided delays). Consider the control systems of Figure 3.1(a) and Figure 3.1(b) where the plant $P(s)$ and the delay operators $\Lambda_u(s)$ and $\Lambda_y(s)$ are given by (3.1,3.2,3.3). Suppose that assumptions A1, A2, and A3 are satisfied. Find a stabilizing LTI causal controller $K_s(s)$ that minimizes the H_2-norm of the transfer function from w to z.

3.3. From standard problem to regulator problem

The transformation is carried out in two stages. In the first stage, the standard problem is converted to a simplified problem having advantageous features of

- a less demanding stability requirement, which only requires the stability of the transfer function from the external input to the external output compared to the internal stability requirement in the original standard problem, and

- invertibility of the plant's (1,2) and (2,1) blocks.

These features allow the second stage, which further transforms the problem to the regulator problem, shown in Figure 3.2.

The subsequent lemmas and theorems provide the details of the transformation. Note that the first stage was developed in [Mir03a] for solving H_2 and H_∞ problems with a single delay. Also note that the first stage applies to the standard problem of Figure 2.1 where delays occur on both sides of the controller.

The idea behind the first stage is to view the delay operators as an additional constraint on the controller. By first obtaining a parametrization of sub-optimal controllers of the delay-free case and then imposing the delays as a constraint, the simplified problem may be obtained.

To this end, we first review the results on the delay-free standard H_2 problem. It is summarized in the following theorem, which is based on Theorem 13.7 of [ZD98] and its proof.

Theorem 3.2 (Delay-free standard H_2 problem). *Consider control system of Figure 2.1 with no delays (i.e. $\Lambda_u = I, \Lambda_y = I$), where the plant P is given by* (3.1) *and the conditions A1, A2, and A3 are satisfied. Let X and Y be the stabilizing solutions of the following two Riccati equations, respectively,*

$$A_X^T X + X A_X - X B_{P2} R_1^{-1} B_{P2}^T X + Q_X = 0, \qquad (3.4)$$

$$A_Y Y + Y A_Y^T - Y C_{P2}^T R_2^{-1} C_{P2} Y + Q_Y = 0, \qquad (3.5)$$

where $R_1 = D_{P12}^T D_{P12}$, $R_2 = D_{P21} D_{P21}^T$, and

$$A_X = (A_P - B_{P2} R_1^{-1} D_{P21}^T C_{P1}), \quad Q_X = C_{P1}(I - D_{P12} R_1^{-1} D_{P12}^T)C_{P1}, \quad (3.6)$$

$$A_Y = (A_P - B_{P1} D_{P21}^T R_2^{-1} C_{P2}), \quad Q_Y = B_{P1}(I - D_{P21}^T R_2^{-1} D_{P21})B_{P1}^T. \quad (3.7)$$

Define

$$F = -R_1^{-1}(B_{P2}^T X + D_{P12}^T C_{P1}), \qquad (3.8)$$

$$L = -(Y C_{P2}^T + B_{P1} D_{P21}^T)R_2^{-1}. \qquad (3.9)$$

Then the family of all stabilizing controllers may be parametrized by

$$K_0(s) = F_\ell(M(s), Q(s)) \qquad (3.10)$$

where

$$M(s) = \left[\begin{array}{c|cc} A_P + B_{P2}F + LC_{P2} & -L & B_{P2} \\ \hline F & 0 & I \\ -C_{P2} & I & 0 \end{array} \right] \qquad (3.11)$$

with $Q \in H_\infty$ but otherwise arbitrary. Furthermore, the squared H_2 norm of the overall transfer function is given by

$$\|F_\ell(P(s), K_0(s))\|_2^2 = \mathrm{tr}(B_{P1}^T X B_{P1}) + \mathrm{tr}(R_1 F Y F^T) + \|R_1^{\frac{1}{2}} Q R_2^{\frac{1}{2}}\|_2^2 \quad (3.12)$$

so that the optimal controller is obtained by setting Q to zero:

$$K_{s,opt}(s) = \left[\begin{array}{c|c} A_P + B_{P2}F + LC_{P2} & -L \\ \hline F & 0 \end{array} \right] \tag{3.13}$$

and the squared optimal H_2-norm is

$$\min_{K_s} \|F_\ell(P(s), K_s(s))\|_2^2 = \mathrm{tr}(B_{P1}^T X B_{P1}) + \mathrm{tr}(R_1 F Y F^T). \tag{3.14}$$

Proof. See [ZD98] □

The next lemma, which is based on the results in [Mir03a], provides the formulas for transforming the standard problem into the simplified problem with a less demanding stability requirement and invertibility of the plant's (1,2) and (2,1) blocks.

Lemma 3.3 (Standard problem to simplified problem). *Consider the standard control system with delays occurring on both sides of the controller, depicted in Figure 2.1. The plant $P(s)$ and the delays operators $\Lambda_y(s), \Lambda_u(s)$ are given by (3.1,3.2,3.3). Suppose that the assumptions A1,A2, and A3 are satisfied. Let X and Y be the stabilizing solutions of the Riccati equations (3.4,3.5), while F and L are given by (3.8,3.9). Then the following hold:*

1. *$K_s(s)$ minimizes $\|F_\ell(P(s), \Lambda_u(s)K_s(s)\Lambda_y(s))\|_2$ if and only if $K_s(s)$ minimizes $\|F_\ell(G(s), \Lambda_u(s)K_s(s)\Lambda_y(s))\|_2$ with*

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & -LR_2^{\frac{1}{2}} & B_{P2} \\ \hline -R_1^{\frac{1}{2}}F & 0 & R_1^{\frac{1}{2}} \\ C_{P2} & R_2^{\frac{1}{2}} & 0 \end{array} \right], \tag{3.15}$$

   where $R_1 = D_{P12}^T D_{P12}$ and $R_2 = D_{P21} D_{P21}^T$,

2. *$K_s(s)$ internally stabilizes the control system of Figure 2.1 if and only if $F_\ell(G(s), \Lambda_u(s)K_s(s)\Lambda_y(s)) \in H_\infty$,*

3. *the squared optimal H_2-norm is*

$$\min_{K_s} \|F_\ell(P(s), \Lambda_u(s)K_s(s)\Lambda_y(s))\|_2^2$$
$$= \min_{K_s} \|F_\ell(P(s), K_s(s))\|_2^2 + \min_{K_s} \|F_\ell(G(s), \Lambda_u(s)K_s(s)\Lambda_y(s))\|_2^2$$
$$= \mathrm{tr}(B_1^T X B_1) + \mathrm{tr}(R_1 F Y F^T) + \min_{K_s} \|F_\ell(G(s), \Lambda_u(s)K_s(s)\Lambda_y(s))\|_2^2. \tag{3.16}$$

Proof. By absorbing the delay operators into the controllers, the problem becomes finding a stabilizing controller that minimizes

$$\|F_\ell(P(s), K_0(s))\|_2 \tag{3.17}$$

over stabilizing $K_0(s)$ under the condition

$$K_0(s) = \Lambda_u(s) K_s(s) \Lambda_y(s). \tag{3.18}$$

Let us for the moment disregard the condition (3.18) and solve the delay-free problem (3.17). Introduce the parametrization of all stabilizing controller for the delay-free problem:

$$K_0(s) = F_\ell(M(s), Q(s)) \tag{3.19}$$

where M is given by (3.11) and Q is an arbitrary stable transfer function. Then, by Theorem 3.2, we have that

$$\|F_\ell(P(s), K_0(s))\|_2^2 = \text{tr}(B_1^T X B_1) + \text{tr}(R_1 F Y F^T) + \|R_1^{\frac{1}{2}} Q(s) R_2^{\frac{1}{2}}\|_2^2. \tag{3.20}$$

Now observe the structure of M in (3.11). Notice that M is invertible and the (1,2) and (2,1) blocks of M are also invertible. Hence, using Lemma 10.4 of [ZDG96] we may invert (3.19) and multiply Q with $R_1^{\frac{1}{2}}$ and $R_2^{\frac{1}{2}}$ to obtain:

$$R_1^{\frac{1}{2}} Q R_2^{\frac{1}{2}} = R_1^{\frac{1}{2}} F_u(M^{-1}, K_0) R_2^{\frac{1}{2}} = F_\ell(G, K_0) \tag{3.21}$$

with $G(s)$ given by (3.15). We see that G also has its (1,2) and (2,1) blocks invertible.

Now let us return to the problem of minimizing

$$\|F_\ell(P(s), \Lambda_u(s) K_s(s) \Lambda_y(s))\|_2$$

by incorporating the condition (3.18). We have that

$$\min_{K_s} \|F_\ell(P(s), \Lambda_u(s) K_s(s) \Lambda_y(s))\|_2^2$$
$$= \min_{K_0 = \Lambda_u K_s \Lambda_y} \|F_\ell(P(s), K_0(s))\|_2^2$$
$$= \text{tr}(B_1^T X B_1) + \text{tr}(R_1 F Y F^T) + \min_{K_0 = \Lambda_u K_s \Lambda_y} \|R_1^{\frac{1}{2}} Q(s) R_2^{\frac{1}{2}}\|_2^2$$
$$= \text{tr}(B_1^T X B_1) + \text{tr}(R_1 F Y F^T) + \min_{K_0 = \Lambda_u K_s \Lambda_y} \|F_\ell(G(s), K_0(s))\|_2^2$$
$$= \text{tr}(B_1^T X B_1) + \text{tr}(R_1 F Y F^T) + \min_{K_s} \|F_\ell(G(s), \Lambda_u(s) K_s(s) \Lambda_y(s))\|_2^2,$$
$$\tag{3.22}$$

which proves statements 1 and 3. Statement 2 follows from the fact that the delay-free control system is internally stable if and only if $Q(s)$, which is equal to
$F_\ell(G(s), \Lambda_u(s) K(s) \Lambda_y(s))$, is stable. $\qquad\square$

Remark 3.4 (Performance loss). The first two terms of the squared optimal H_2-norm (3.22) is the squared optimal norm for the delay-free case (see Theorem 3.2). Thus, the last term may be interpreted as the performance loss due to the presence of delay.

Using Lemma 3.3, the standard problem may be reduced to the simplified problem. By employing loop shifting arguments like in the proof of Lemma 2.5, the simplified problem may be further reduced to the regulator problem of Figure 3.2 with T_1, T_2 replaced by G_{11}, G_{12} for the input delays case and G_{11}^T, G_{21}^T for the output delays case. The subsequent two lemmas corresponding with the input delay case and the output delay case summarize the complete transformation from the standard problem to the regulator problem.

Lemma 3.5 (Input delays). *Consider the problem of minimizing*

$$\|F_\ell(P(s), \Lambda_u(s) K_s(s))\|_2 \tag{3.23}$$

over stabilizing, causal K_s, where $P(s)$ and $\Lambda_u(s)$ are given by (3.1) and (3.3), respectively. It is equivalent to minimizing

$$\|G_{11}(s) + G_{12}(s)\Lambda_u(s)K(s)\|_2 \tag{3.24}$$

over causal K, with $G(s)$ given by (3.15), where there is a proper bijection between K and K_s. The bijection is governed by by the following two equations via the variable $\tilde{K}_s$:

$$K_s(s) = (I - \tilde{K}_s(s)\Phi_{22}(s))^{-1}\tilde{K}_s(s), \tag{3.25}$$

$$\tilde{K}_s(s) = K(s)\left(I + G_{21}^{-1}(s)\tilde{G}_{22}(s)K(s)\right)^{-1} G_{21}^{-1}(s) \tag{3.26}$$

$$= F_\ell(V_u(s), K(s)),$$

where

$$\Phi_{22}(s) = \tilde{G}_{22}(s) - G_{22}(s)\Lambda_u(s), \tag{3.27}$$

$$\tilde{G}_{22}(s) = C_{P2}(sI - A_P)^{-1}\tilde{B}_{P2}, \tag{3.28}$$

$$\tilde{B}_{P2} = \begin{bmatrix} e^{-A_P h_{u1}} B_{P2,1} & \dots & e^{-A_P h_{up}} B_{P2,p} \end{bmatrix} \tag{3.29}$$

$$V_u(s) = \begin{bmatrix} 0 & I \\ G_{21}^{-1}(s) & -G_{21}^{-1}(s)\tilde{G}_{22}(s) \end{bmatrix}. \tag{3.30}$$

Moreover, the squared optimal H_2-norm is given by

$$\begin{aligned} &\min_{K_s} \|F_\ell(P(s), \Lambda_u(s)K_s(s))\|_2^2 = \\ &\mathrm{tr}(B_1^T X B_1) + \mathrm{tr}(R_1 FY F^T) + \min_K \|G_{11}(s) + G_{12}(s)\Lambda_u(s)K(s)\|_2^2 \end{aligned} \tag{3.31}$$

where X and Y are the stabilizing solution of the Riccati equations (3.4,3.5), F is given by (3.8), and $R_1 = D_{P12}^T D_{P12}$.
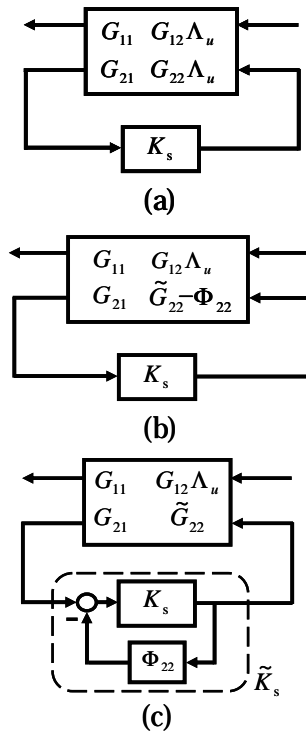
Figure 3.3.: Removing the delay from the (2,2) block.

Proof. Consider the input delay configuration of Figure 3.1(a). By Lemma 3.3, the problem of minimizing $\|F_\ell(P(s), \Lambda_u(s)K_s(s))\|_2$ over stabilizing, causal K_s is equivalent to the simplified problem of minimizing

$$\|F_\ell(G(s), \Lambda_u(s)K_s(s))\|_2$$

over causal K_s without the internal stability requirement. The configuration corresponding to the simplified problem with the delay operator absorbed into the plant is shown in Figure 3.3(a). Using (3.27) and block diagram manipulation similar to the one used in the proof of Lemma 2.5, the configuration of Figure 3.3(a) is transformed to the one of Figure 3.3(c), i.e.

$$F_\ell(G(s), \Lambda_u(s)K_s(s)) = F_\ell(\begin{bmatrix} G_{11}(s) & G_{12}(s)\Lambda_u(s) \\ G_{21}(s) & \tilde{G}_{22}(s) \end{bmatrix}, \tilde{K}_s(s)) \tag{3.32}$$

where

$$\tilde{K}_s(s) = (I + K_s(s)\Phi_{22}(s))^{-1}K_s(s). \tag{3.33}$$

It follows that K_s may be inferred from $\tilde{K}_s$ using (3.25). The proper bijection between $\tilde{K}_s$ and K_s is proved in Lemma 2.4. Let us return to (3.32). We have that

$$\begin{aligned} F_\ell(G(s), \Lambda_u(s)K_s(s)) &= F_\ell\left(\begin{bmatrix} G_{11}(s) & G_{12}(s)\Lambda_u(s) \\ G_{21}(s) & \tilde{G}_{22}(s) \end{bmatrix}, \tilde{K}_s(s)\right) \\ &= G_{11}(s) + G_{12}(s)\Lambda_u(s)\underbrace{\tilde{K}_s(s)(I - \tilde{G}_{22}(s)\tilde{K}_s(s))^{-1}G_{21}(s)}_{=:K(s)}. \end{aligned} \tag{3.34}$$

Hence, by defining

$$K(s) = \tilde{K}_s(s)(I - \tilde{G}_{22}(s)\tilde{K}_s(s))^{-1}G_{21}(s) \tag{3.35}$$

we have that

$$F_\ell(G(s), \Lambda_u(s)K_s(s)) = G_{11}(s) + G_{12}(s)\Lambda_u(s)K(s). \tag{3.36}$$

In addition, since $G_{21}(s)$ is invertible, the relation between $\tilde{K}_s(s)$ and $K(s)$ is a bijection, where $\tilde{K}(s)$ may be obtained using (3.26). Using similar arguments as in Lemma 2.4 and the fact that $\tilde{G}_{22}$ is strictly proper, it may be shown that $\tilde{K}_s(s)$ is proper if and only if $K(s)$ is proper. The optimal H_2-norm follows from statement 3 of Lemma 3.3. $\square$

Lemma 3.6 (Output delays). *Consider the problem of minimizing*

$$\|F_\ell(P(s), K_s(s)\Lambda_y(s))\|_2 \tag{3.37}$$

over stabilizing, causal K_s, where $P(s)$ and $\Lambda_y(s)$ are given by (3.1) and (3.2), respectively. It is equivalent to minimizing

$$\|G_{11}(s) + K(s)\Lambda_y(s)G_{21}(s)\|_2 \tag{3.38}$$

over causal K, with $G(s)$ given by (3.15), where there is a proper bijection between K and K_s. The bijection is governed by the following two equations via the variable $\tilde{K}_s$:

$$K_s(s) = (I - \tilde{K}_s(s)\Phi_{22}(s))^{-1}\tilde{K}_s(s), \tag{3.39}$$

$$\tilde{K}_s(s) = G_{12}^{-1}(s)K(s)\left(I + \tilde{G}_{22}(s)G_{12}^{-1}(s)K(s)\right)^{-1}, \tag{3.40}$$

$$= F_\ell(V_y(s), K(s)),$$

where

$$\Phi_{22}(s) = \tilde{G}_{22}(s) - \Lambda_y(s)G_{22}(s), \tag{3.41}$$

$$\tilde{G}_{22}(s) = \tilde{C}_{P2}(sI - A_P)^{-1}B_{P2}, \tag{3.42}$$

$$\tilde{C}_{P2} = \text{col}\left[C_{P2,1}e^{-A_P h_{y1}}, \dots, C_{P2,m}e^{-A_P h_{ym}}\right] \tag{3.43}$$

$$V_y(s) = \begin{bmatrix} 0 & G_{12}^{-1}(s) \\ I & -\tilde{G}_{22}(s)G_{12}^{-1}(s) \end{bmatrix}. \tag{3.44}$$

Moreover, the squared optimal H_2-norm is given by

$$\begin{aligned}
\min_{K_s} \|F_\ell(P(s), K_s(s)\Lambda_y(s))\|_2^2 = \\
\text{tr}(B_1^T X B_1) + \text{tr}(R_1 F Y F^T) + \min_K \|G_{11}(s) + K\Lambda_y(s)G_{21}(s)\|_2^2,
\end{aligned} \tag{3.45}$$

where X and Y are the stabilizing solution of the Riccati equations (3.4,3.5), F is given by (3.8), and $R_1 = D_{P12}^T D_{P12}$.

Proof. This lemma is the dual of Lemma 3.5 and may be proved using similar arguments. $\qquad\square$

Remark 3.7 (Filtering problem). The equivalent problem of minimizing (3.38) for the output delays case is not exactly a regulator problem of Figure 3.2. It is actually a filtering problem with measurement delays. However, since transposition of a transfer function does not change its H_2-norm, we have that

$$\min_K \|G_{11}(s) + K(s)\Lambda_y(s)G_{21}(s)\|_2 = \min_{K^T} \|G_{11}^T(s) + G_{21}^T(s)\Lambda_y(s)K^T(s)\|_2. \tag{3.46}$$

The right hand side is clearly a regulator problem.

3.4. From the regulator problem to LQR problem with multiple input delays

The previous section shows that the standard H_2 problem with delays occurring on one side of the controller (Figure 3.1(a),(b)) may be reduced to the regulator problem of Figure 3.2. In this section, we show that the regulator problem, when viewed in time domain, may be recast to an LQR problem with multiple input delays.

First we formulate the regulator problem of Figure 3.2 more precisely. We assume that the LTI systems T_1 and T_2 have the joint realization

$$\begin{bmatrix} T_1(s) & T_2(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_2 \end{array} \right]. \tag{3.47}$$

Without loss of generality, we may also assume that the delays in the delay operator Λ are in ascending order according to their magnitude so that it may be written in the form

$$\begin{aligned} \Lambda(s) &= \operatorname{diag}\left(e^{-sh_0} I_0, e^{-sh_1} I_1, ..., e^{-sh_N} I_N \right), \tag{3.48} \\ 0 &= h_0 < h_1 < \cdots < h_N. \end{aligned}$$

Note that I_0 may be empty which means that there are no non-delayed channels.

Problem 3.8 (H_2 regulator problem). Consider the control systems of Figure 3.2 where the systems $T_1(s)$, $T_2(s)$, and the delay operator $\Lambda(s)$ are given by (3.47,3.48). Find an LTI causal controller $K(s)$ such that

$$\| T_1(s) + T_2(s)\Lambda(s)K(s) \|_2$$

is minimized.

Next we show that Problem 3.8, when viewed in time domain, is equivalent to an LQR problem with multiple input delays. This result is stated in the following lemma.

Lemma 3.9 (Regulator problem to LQR problem with delays). *Consider Problem 3.8. Denote the j-th column of the controller K by K_j. Partition T_2 and K_j according to the delays:*

$$\begin{aligned} \begin{bmatrix} T_1(s) & T_2(s) \end{bmatrix} &= \begin{bmatrix} T_1(s) & \begin{bmatrix} T_{2,0}(s) & \cdots & T_{2,N}(s) \end{bmatrix} \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A & B_1 & \begin{bmatrix} B_{2,0} & \cdots & B_{2,N} \end{bmatrix} \\ \hline C_1 & 0 & \begin{bmatrix} D_{2,0} & \cdots & D_{2,N} \end{bmatrix} \end{array} \right] \end{aligned} \tag{3.49}$$

$$K_j(t) = \begin{bmatrix} K_{j,0}(t) \\ \vdots \\ K_{j,N}(t) \end{bmatrix}. \tag{3.50}$$

Here $T_{2,k}(s)$, $B_{2,k}$, $D_{2,k}$ and $K_{j,k}(t)$ are the respective parts of $T_2(s)$, B_2, D_2, and $K_j(t)$ that correspond to the delay h_k.

Then the optimal $K_j(t)$ is given by

$$K_{j,opt}(t) = \arg \min_{K_j} \int_0^\infty \|C_1 x(t) + \sum_{k=0}^N D_{2,k} K_{j,k}(t-h_k)\|_2^2 dt, \qquad (3.51)$$

where the state x obeys the state equation

$$\dot{x}(t) = Ax(t) + \sum_{k=0}^N B_{2,k} K_{j,k}(t-h_k), \quad x(0) = B_{1,j} \qquad (3.52)$$

with $B_{1,j}$ is the j-th column of B_1.

Proof. Using Parseval's theorem, we may translate the the H_2 regulator problem to an optimization in time domain:

$$K_{\mathrm{opt}}(t) = \arg \min_K \|T_1(t) + T_2(t) * \Lambda(t) * K(t)\|_2 \qquad (3.53)$$

where $\Lambda(t) = \mathrm{diag}(\delta(t)I_0, \delta(t-h_1)I_1, \ldots, \delta(t-h_N)I_N)$ and the $'*'$ operator denotes the convolution operator. It is straightforward to show that we may find any individual column of $K_{\mathrm{opt}}(t)$ independently and it is given by

$$K_{j,\mathrm{opt}}(t) = \arg \min_{K_j} \|T_{1,j}(t) + T_2(t) * \Lambda(t) * K_j(t)\|_2 \qquad (3.54)$$

where $K_j(t)$ and $T_{1,j}(t)$ are the j-th column of $K(t)$ and $T_1(t)$, respectively. Here, the convolution of the delay operator and the controller is given by

$$\Lambda(t) * K_j(t) = \begin{bmatrix} K_{j,0}(t) \\ K_{j,1}(t-h_1) \\ \vdots \\ K_{j,N}(t-h_N) \end{bmatrix}. \qquad (3.55)$$

From (3.54), it is apparent that we may view the quantity inside the norm brackets as the output of the system $\begin{bmatrix} T_{1,j}(s) & T_2(s) \end{bmatrix}$ with $\mathrm{col}[\delta(t), \Lambda(t) * K_j(t)]$ as the input. Moreover, the realization (3.49) allows us to write down the state-space equation for this system, which is given by (3.52), and the objective (3.54) becomes minimizing the criterion function (3.51). We see that the system description (3.52) and the associated criterion function (3.51) constitute an LQR control problem with multiple input delays. $\square$

3.5. Solution to the LQR problem with multiple input delays

In the previous section we show that solving the regulator problem of Figure 3.2 amounts to solving the LQR problem with input delays of the form

(3.52,3.51). In this section we treat the LQR problem, which is formally formulated in what follows.

Problem 3.10 (LQR problem with multiple input delays). Let a system be defined by the state equation

$$\dot{x}(t) = Ax(t) + \sum_{k=0}^{N} B_{2,k} u_k(t - h_k), \quad x(0) = x_0$$
$$z(t) = C_1 x(t) + \sum_{k=0}^{N} D_{2,k} u_k(t - h_k)$$

(3.56)

where $0 = h_0 < h_1 < \ldots < h_N$ and the initial condition x_0 is given but arbitrary. Suppose the system matrices satisfy the following conditions:
A5 (C_1, A) and (A, B_2) are detectable and stabilizable, respectively;
A6 $R = D_2^T D_2 > 0$,
A7 $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_2 \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.
Here the matrices B_2 and D_2 are given by

$$B_2 = \begin{bmatrix} B_{2,0} & \ldots & B_{2,N} \end{bmatrix}$$
$$D_2 = \begin{bmatrix} D_{2,0} & \ldots & D_{2,N} \end{bmatrix}.$$

(3.57)

Find an optimal control law $u(t) = \text{col}[u_0(t), \ldots, u_N(t)]$ such that criterion function

$$J(x_0, u) = \int_0^\infty \|z(t)\|_2^2 dt$$

(3.58)

is minimized.

To solve the LQR problem with multiple input delays, we show that we may reduce the optimal control problem (3.56,3.58) to a series of standard LQR problems. To this end, first note that due to the delays, each input u_k is active only for $t \geq h_k$. For the case with two nonzero delays, this fact is depicted in Figure 3.4(a). Furthermore, from the figure, it is clear that if we restrict ourselves to a time region between the delays, the input signal in that time region is delay free. This fact gives the motivation to transform the inputs $u_k(t), k = 0, \ldots, N$ to a new set of inputs, each of which is active only in a particular time region delimited by either two adjacent delays or the largest delay and $t = \infty$.

These new inputs, denoted by $\phi_k(t), k = 0, \cdots, N$ is constructed by stacking the delayed original inputs that are active in a particular time region as depicted in Figure 3.4(b) (shown for two nonzero delays case). Mathematically, the new input corresponding to the time region $t \in [h_k, h_{k+1}]$, where
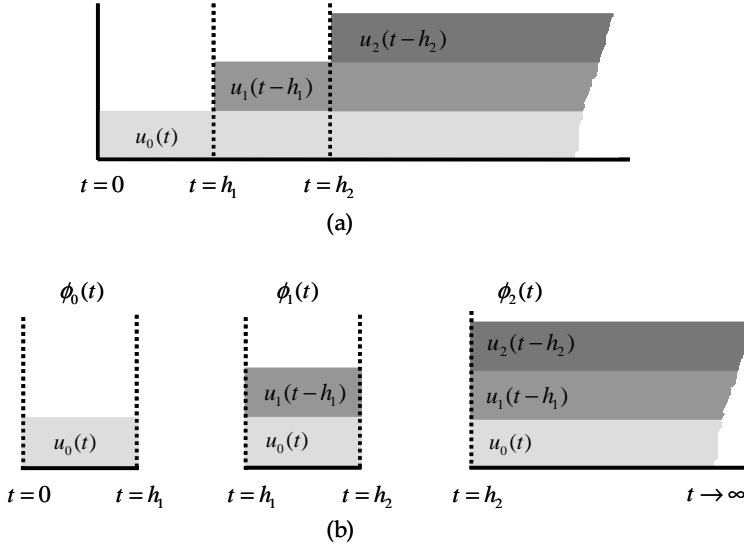
(a)

(b)

Figure 3.4.: Constructing the new inputs $\phi_k(t)$ for the case with 2 nonzero delays.

$k = 0, \ldots, N-1$, is defined as

$$\phi_k(t) = \begin{bmatrix} u_0(t) \\ u_1(t - h_1) \\ \vdots \\ u_k(t - h_k) \end{bmatrix} \left[\mathbb{1}(t - h_k) - \mathbb{1}(t - h_{k+1}) \right], \tag{3.59}$$

while for the time region $t \in [h_N, \infty]$ the new input is

$$\phi_N(t) = \begin{bmatrix} u_0(t) \\ u_1(t - h_1) \\ \vdots \\ u_N(t - h_N) \end{bmatrix} \mathbb{1}(t - h_N). \tag{3.60}$$

By transforming the input according to (3.59,3.60), the LQR problem associated with the equations (3.56,3.58) is transformed as well:

$$\dot{x}(t) = Ax(t) + \sum_{k=0}^{N} \bar{B}_k \phi_k(t), \ x(0) = x_0, \tag{3.61}$$

$$J(x_0, \phi_0, ..., \phi_N) = \int_0^\infty \|C_1 x(t) + \sum_{k=0}^{N} \bar{D}_k \phi_k(t)\|_2^2 dt, \tag{3.62}$$

where

$$\bar{B}_k := \begin{bmatrix} B_{2,0} & \dots & B_{2,k} \end{bmatrix}, \qquad \bar{D}_k := \begin{bmatrix} D_{2,0} & \dots & D_{2,k} \end{bmatrix}. \qquad (3.63)$$

An important feature of the transformed LQR problem (3.61,3.62) is that in a particular time region delimited by either two adjacent delays or the largest delay and $t = \infty$, only one input $\phi_k(t)$ is active. For example, on the time region $t \in [h_1, h_2]$ only $\phi_1(t)$ is active. To exploit this feature for solving the LQR problem with delays, we need another ingredient. We require a standard result in optimal control theory, namely the principle of optimality (see Theorem A.1 in Section A.1 of the Appendix for details).

One implication of the principle is the following. Suppose we have an optimal control problem over the time region $t \in [t_0, t_e]$. Then we may solve the problem over $t \in [t_1, t_e], t_1 > t_0$ independent of what happens during $t \in [t_0, t_1]$, provided that we start with the optimal state at $t = t_1$.

Let us apply this principle to the transformed LQR problem (3.61,3.62) for

$$t_0 = 0, \ t_1 = h_N, \ t_e = \infty.$$

On $t \in [h_N, \infty]$, only ϕ_N is active. Assume that $x_{\text{opt}}(t), t \leq h_N$ is known and apply the principle of optimality. We have that

$$\min_{\phi_0,\cdots,\phi_N} J(x_0, \phi_0, \cdots, \phi_N) = \min_{\phi_0,\cdots,\phi_N} \int_0^\infty \|C_1 x(t) + \sum_{k=0}^N \bar{D}_k \phi_k(t)\|_2^2 dt$$

$$= \int_0^{h_N} \|C_1 x_{\text{opt}}(t) + \sum_{k=0}^{N-1} \bar{D}_k \phi_{k,\text{opt}}(t)\|_2^2 dt + \min_{\phi_N} J_{[h_N,\infty]}(x_{\text{opt}}(h_N), \phi_N)$$

$$(3.64)$$

where

$$J_{[h_N,\infty]}(x(h_N), \phi_N) = \int_{h_N}^\infty \|C_1 x(t; x(h_N)) + D_2 \phi_N(t)\|_2^2 dt. \qquad (3.65)$$

Furthermore, the state space equation (3.61) in the time region $t \in [h_N, \infty]$ becomes

$$\dot{x}(t) = Ax(t) + B_2 \phi_N(t),$$
$$x(h_N) = x_N(x_0, \phi_0, \cdots, \phi_{N-1}). \qquad (3.66)$$

The equations (3.66) with $x(h_N) = x_{\text{opt}}(h_N)$ and (3.65) constitute a standard infinite horizon LQR problem. By solving this problem, the optimal input from $t = h_N$ onwards, namely ϕ_N, is completely determined as a function of $x_{\text{opt}}(h_N)$. Moreover, it is well known that the optimal cost $J_{[h_N,\infty],\text{opt}}$ is quadratic in the initial state $x_{\text{opt}}(h_N)$:

$$\min_{\phi_N} J_{[h_N,\infty]}(x_{\text{opt}}(h_N), \phi_N) = x_{\text{opt}}(h_N)^T M_N x_{\text{opt}}(h_N), \qquad (3.67)$$

for a certain constant matrix $M_N \geq 0$. Thus substituting (3.67) into (3.64), we obtain

$$
\min_{\phi_0, \cdots, \phi_N} J(x_0, \phi_0, \cdots, \phi_N)
$$
$$
= \int_0^{h_N} \|C_1 x_{\text{opt}}(t) + \sum_{k=0}^{N-1} \bar{D}_k \phi_{k,\text{opt}}(t)\|_2^2 dt + x_{\text{opt}}(h_N)^T M_N x_{\text{opt}}(h_N)
$$
$$
= \min_{\phi_0, \cdots, \phi_{N-1}} \left(x(h_N)^T M_N x(h_N) + \int_0^{h_N} \|C_1 x(t) + \sum_{k=0}^{N-1} \bar{D}_k \phi_k(t)\|_2^2 dt \right).
$$
(3.68)

This means that the infinite horizon problem (3.62) is reduced to a finite horizon problem (3.68) over $t \in [0, h_N]$, in which ϕ_N no longer plays a role. We may then continue to apply the principle of optimality for $t_1 = h_{N-1}$ and $t_e = h_N$, which results in a finite horizon LQR problem over the time region $t \in [h_{N-1}, h_N]$:

$$
\dot{x}(t) = Ax(t) + \bar{B}_{N-1}\phi_{N-1}(t), \ x(h_{N-1}) = x_{N-1}(x_0, \phi_0, \cdots, \phi_{N-2}), \quad (3.69)
$$

$$
\min_{\phi_{N-1}} \left(x(h_N)^T M_N x(h_N) + \int_{h_{N-1}}^{h_N} \|C_1 x(t; x(h_{N-1})) + \bar{D}_{N-1}\phi_{N-1}(t)\|_2^2 dt \right).
$$
(3.70)

By solving the above LQR problem using standard LQR theory, we obtain the optimal input on the time region $t \in [h_{N-1}, h_N]$, namely ϕ_{N-1}, as a function of the optimal state at $t = h_{N-1}$. Furthermore, we may express the cost contribution of the time region $t \in [h_{N-1}, \infty]$ of the original LQR problem (3.61,3.62) as a quadratic function of the optimal state at $t = h_{N-1}$, given by

$$
x_{\text{opt}}(h_{N-1})^T M_{N-1} x_{\text{opt}}(h_{N-1}),
$$

for a certain constant matrix $M_{N-1} \geq 0$. This allows us to further shrink the optimization time interval of the LQR problem (3.61,3.62) to $t \in [0, h_{N-1}]$, in which ϕ_{N-1} and ϕ_N no longer play a role.

By continuing in this fashion we may obtain the optimal input by solving the optimal control problem backward in time, time-region by time-region. The optimal input for each time region is expressed as a function of the optimal initial state of time region. Since we know the optimal state at $t = 0$, we may then move forward in time to compute the optimal input for each time region.

Having computed the optimal new inputs ϕ_k using the algorithm described earlier, we may recover the optimal original input $u(t)$, which is given by

$$
u_{\text{opt}}(t) = \Lambda^{\sim}(t) * \phi_{\text{opt}}(t) \tag{3.71}
$$

where

$$\Lambda^{\sim}(t) = \begin{bmatrix} \delta(t)I_0 & & & \\ & \delta(t+h_1)I_1 & & \\ & & \ddots & \\ & & & \delta(t+h_N)I_N \end{bmatrix} \tag{3.72}$$

$$\phi_{\text{opt}}(t) = \begin{bmatrix} \phi_{0,\text{opt}}(t) \\ 0_1 \\ \vdots \\ 0_N \end{bmatrix} + \begin{bmatrix} \phi_{1,\text{opt}}(t) \\ 0_2 \\ \vdots \\ 0_N \end{bmatrix} + \cdots \quad + \begin{bmatrix} \phi_{N-1,\text{opt}}(t) \\ 0_N \end{bmatrix} + \phi_{N,\text{opt}}(t).$$

$$\tag{3.73}$$

Here I_k and 0_k, with $k = 1, \cdots, N$, are identity matrices and zero column vectors having dimension equal to the dimension of $u_k(t)$. Note that $u_{\text{opt}}(t)$ is causal since $\phi_{k,\text{opt}}$ is identically zero outside $t \in [h_k, h_{k+1}]$.

From the discussion, it is evident that the solution to the LQR problem with multiple input delays (3.56,3.58) amounts to solving standard regional LQR problems for the time regions $t \in [h_N, \infty], t \in [h_{N-1}, h_N], \ldots, t \in [0, h_1]$. The first problem ($t \in [h_N, \infty]$) is an infinite horizon problem, while the rest are finite horizon problems.

The subsequent two lemmas provides the formulas of the solution of the regional problems. These problems are standard non-delayed LQR problems and may be solved using existing LQR theory, which is reviewed in Section A.3 of the Appendix. The first lemma deals with the infinite horizon problem of the system (3.66) with the criterion function (3.65) corresponding to the time region $t \in [h_N, \infty]$.

Lemma 3.11 (Infinite horizon regional LQR problem). *Consider the system*

$$\dot{x}(t) = Ax(t) + B_2\phi_N(t), \qquad x(h_N) = x_N, \tag{3.74}$$

with x_N known but arbitrary, and the criterion function

$$\min_{\phi_N} J_{[h_N,\infty]}(x(h_N), \phi_N) = \min_{\phi_N} \int_{h_N}^{\infty} \|C_1 x(t) + D_2\phi_N(t)\|_2^2 dt, \tag{3.75}$$

where the matrices A, B_2, C_1, and D_2 are as in (3.56,3.57) and h_N is the largest delay in (3.56). Suppose that the assumptions A5, A6, and A7 on page 44 are satisfied. Let M_N be the stabilizing solution of the following Riccati equation:

$$Q_N + A_N^T M_N + M_N A_N - M_N B_2 R_N^{-1} B_2^T M_N = 0 \tag{3.76}$$

where

$$A_N = A - B_2 R_N^{-1} D_2^T C_1, \quad R_N = D_2^T D_2, \quad Q_N = C_1^T (I - D_2 R_N^{-1} D_2^T) C_1. \tag{3.77}$$

Define

$$F_N = -R_N^{-1} \left(B_2^T M_N + D_2^T C_1 \right),$$
$$S_N = A + B_2 F_N \tag{3.78}$$

then the optimal control on $t \in [h_N, \infty]$ is given by the state feedback

$$\phi_{N,opt}(t) = F_N x(t) \mathbb{1}(t - h_N), \tag{3.79}$$

which may also be expressed as the impulse response of the transfer function

$$\phi_{N,opt}(s) = e^{-sh_N} \left[\begin{array}{c|c} S_N & x_N \\ \hline F_N & 0 \end{array} \right]. \tag{3.80}$$

Moreover, the optimal cost is

$$J_{[h_N,\infty],opt}(x,_N, \phi_N) = x_N^T M_N x_N. \tag{3.81}$$

Proof. The Lemma is an application of the results in Appendix A.3.2 to the infinite horizon LQR problem (3.74,3.75). □

The second lemma solves the regional LQR problem for the time regions for the time regions $t \in [h_{N-1}, h_N], \ldots, t \in [0, h_1]$.

Lemma 3.12 (Finite horizon regional LQR problem). *Define*

$$\bar{B}_k = \begin{bmatrix} B_{2,0} & \ldots & B_{2,k} \end{bmatrix}, \qquad \bar{D}_k = \begin{bmatrix} D_{2,0} & \ldots & D_{2,k} \end{bmatrix}, \tag{3.82}$$

where $B_{2,k}$ and $D_{2,k}$, with $k = 0, \ldots, N$, are as in (3.56). Let the matrices A and C_1 be as in (3.56). Consider the system

$$\dot{x}(t) = Ax(t) + B_k\phi_k(t), \qquad x(h_k) = x_k \tag{3.83}$$

and the objective of minimizing the criterion function

$$\min_{\phi_k} J_{[h_k,h_{k+1}]}(x(h_k), \phi_k)$$
$$= \min_{\phi_k} \ x(h_{k+1})^T M_{k+1} x(h_{k+1}) + \int_{h_k}^{h_{k+1}} \|C_1 x(t) + \bar{D}_k \phi_k(t)\|^2 dt \tag{3.84}$$

where x_k and $M_{k+1} \geq 0$ are known but arbitrary. The solution to the LQR problem (3.83,3.84) is given in what follows. Define

$$L_k = h_{k+1} - h_k, \qquad R_k = \bar{D}_k^T D_k, \tag{3.85}$$

$$Q_k = C_1^T (I - \bar{D}_k R_k^{-1} \bar{D}_k^T) C_1 \geq 0, \qquad F_k = \begin{bmatrix} -R_k^{-1} \bar{D}_k^T C_1 & R_k^{-1} \bar{B}_k^T \end{bmatrix}. \tag{3.86}$$

$$S_k = \begin{bmatrix} (A - \bar{B}_k R_k^{-1} \bar{D}_k^T C_1) & \bar{B}_k R_k^{-1} \bar{B}_k^T \\ Q_k & -(A - B_k R_k^{-1} \bar{D}_k^T C_1)^T \end{bmatrix} \qquad (3.87)$$

$$\Sigma_k(t) = \begin{bmatrix} \Sigma_{k,11}(t) & \Sigma_{k,12}(t) \\ \Sigma_{k,21}(t) & \Sigma_{k,22}(t) \end{bmatrix} = e^{S_k t} \qquad (3.88)$$

$$\tilde{\Sigma}_k(t) = \begin{bmatrix} \Sigma_{k,22}(t) & \Sigma_{k,21}(t) \\ -\Sigma_{k,12}(t) & -\Sigma_{k,11}(t) \end{bmatrix} \qquad (3.89)$$

$$\Gamma_k(t) = \Sigma_{k,11}(t) + \Sigma_{k,12}(t) C_\ell(\tilde{\Sigma}_k(L_k), M_{k+1}) \qquad (3.90)$$

$$\Xi_k(t) = \Sigma_{k,21}(t) + \Sigma_{k,22}(t) C_\ell(\tilde{\Sigma}_k(L_k), M_{k+1}) \qquad (3.91)$$

$$P_k = \begin{bmatrix} I \\ C_\ell(\tilde{\Sigma}_k(L_k), M_{k+1}). \end{bmatrix} \qquad (3.92)$$

Here

$$
\begin{aligned}
C_\ell(\tilde{\Sigma}_k, M_{k+1}) &= -\left(\tilde{\Sigma}_{k,11} - M_{k+1}\tilde{\Sigma}_{k,21}\right)^{-1} \left(\tilde{\Sigma}_{k,12} - M_{k+1}\tilde{\Sigma}_{k,22}\right) \\
&= -\left(\Sigma_{k,22} + M_{k+1}\Sigma_{k,12}\right)^{-1} \left(\Sigma_{k,21} + M_{k+1}\Sigma_{k,11}\right)
\end{aligned}
\qquad (3.93)
$$

denotes the left homographic transformation. Then the optimal input that minimizes the criterion function (3.84) *is given by*

$$\phi_{k,opt}(t) = F_k e^{S_k t} P_k x_k (\mathbb{1}(t - h_{k+1}) - \mathbb{1}(t - h_k)), \qquad (3.94)$$

which may also be expressed as the impulse response of the transfer function

$$\phi_{k,opt}(s) = e^{-s h_k} \tau_{L_k} \left(\left[\begin{array}{c|c} S_k & P_k x_k \\ \hline F_k & 0 \end{array} \right] \right). \qquad (3.95)$$

Moreover, the optimal cost is given by

$$J_{[h_k, h_{k+1}], opt}(x(h_k), \phi_k) = x(h_k)^T M_k x(h_k), \qquad (3.96)$$

where

$$M_k = -\Xi_k(0) \qquad (3.97)$$

and the optimal final state is

$$x_{opt}(h_{k+1}) = \Gamma_k(L_k) x(h_k). \qquad (3.98)$$

Proof. The Lemma is obtained by applying the results of Appendix A.3.1 to the LQR problem (3.83,3.84). $\qquad \square$

Now we are in the position to formulate an algorithm that solves the LQR problem with delays (3.56,3.58). The algorithm is formally stated in the following theorem.

Theorem 3.13 (Algorithm for the LQR problem with delays). *Consider the LQR problem with multiple input delays (Problem 3.10) with the system equation (3.56) and the criterion function (3.58). Suppose that the assumptions A5, A6, and A7 on page 44 are satisfied. Define the new input ϕ_k, $k = 0, \ldots, N$ as in (3.59,3.60). Also define $\bar{B}_k$ and $\bar{D}_k$ as in (3.82). Then (3.56) and (3.58) become*

$$\dot{x}(t) = Ax(t) + \sum_{k=0}^{N} \bar{B}_k \phi_k(t), \ x(0) = x_0, \tag{3.99}$$

$$J(x_0, \phi_0, ..., \phi_N) = \int_0^\infty \|C_1 x(t) + \sum_{k=0}^{N} \bar{D}_k \phi_k(t)\|_2^2 dt. \tag{3.100}$$

The optimal input may then be computed as followed.

1. *Solve the infinite horizon LQR problem*

$$\dot{x}(t) = Ax(t) + B_2 \phi_N(t), \ \ x(h_N) = x_N, \tag{3.101}$$

$$\min_{\phi_N} \int_{h_N}^\infty \|C_1 x(t; x(h_N)) + D_2 \phi_N(t)\|_2^2 dt. \tag{3.102}$$

   Note that the optimal state at $t = h_N$, denoted by $x_N = x_{opt}(h_N)$, is not yet known. By Lemma 3.11, the optimal cost is given by

$$x_{opt}(h_N)^T M_N x_{opt}(h_N), \tag{3.103}$$

   where M_N is the stabilizing solution of the Riccati equation (3.76). The optimal input for this region is the impulse response of the transfer function:

$$\phi_{N,opt}(s) = e^{-sh_N} F_N (sI - S_N)^{-1} x_N \tag{3.104}$$

   given by (3.80), where S_N and F_N are given by (3.78). Next, set $k := N-1$.

2. *Solve the finite horizon LQR problem*

$$\dot{x}(t) = Ax(t) + \bar{B}_k \phi_k(t), \ \ x(h_k) = x_k, \tag{3.105}$$

$$\min_{\phi_k} \ x(h_{k+1})^T M_{k+1} x(h_{k+1}) + \int_{h_k}^{h_{k+1}} \|C_1 x(t) + \bar{D}_k \phi_k(t)\|^2 dt. \tag{3.106}$$

   Again $x_k = x_{opt}(h_k)$ is not yet known. By Lemma 3.12, the optimal cost is given by

$$x_{opt}(h_k)^T M_k x_{opt}(h_k) \tag{3.107}$$

   where M_k may be computed using (3.97). Then the optimal $\phi_k(t)$ is the impulse response of the transfer function (3.95), :

$$\phi_{k,opt}(s) = e^{-sh_k} \tau_{L_k} \left(F_k (sI - S_k)^{-1} P_k x_k \right), \tag{3.108}$$

   where P_k, S_k, and F_k are given by (3.92), (3.87), and (3.86), respectively, while $L_k = h_{k+1} - h_k$. Next, set $k := k - 1$.

3. *If $k > 0$ then repeat step 2, otherwise go to the next step. At this stage M_1, M_2, $\cdots$, M_N are known.*

4. *Solve the LQR problem of the system (3.105) for $k = 0$ with the criterion function (3.106) for the case $x(0) = x_0$. By Lemma 3.12, the optimal control $\phi_{0,opt}(t)$ is the impulse response of the transfer function (3.108) with $k = 0$, which is a function of x_0. Furthermore, the optimal cost, which is also the optimal cost of the LQR problem with input delays (3.56,3.58), is given by*

$$J_{[0,h_1],opt}(x(0), \phi_0) = x_0^T M_0 x_0 \qquad (3.109)$$

   where M_0 may be computed using (3.97) for $k = 0$. Compute the optimal state at $t = h_1$, denoted by x_1, using (3.98) for $k = 0$.

5. *Compute the optimal control $\phi_{k+1,opt}(t)$ by substituting $x(h_{k+1}) = x_{k+1}$ into (3.108). Next, compute the optimal state at $t = h_{k+2}$, denoted by x_{k+2}, using (3.98). Next, set $k := k + 1$.*

6. *If $k < N$ repeat step 5 otherwise do the following. Substitute the optimal state at $t = h_N$, denoted by x_N to (3.104) to obtain $\phi_{N,opt}(t)$. Finally, substitute $\phi_{k,opt}, k = 0, ..., N$ to (3.71) to obtain u_{opt}.*

3.6. Constructing the optimal controller for the regulator problem

Let us return to the regulator problem (Problem 3.8). By Lemma 3.9, solving the regulator problem amounts to solving M LQR problems of the form

$$\dot{x}(t) = Ax(t) + \sum_{k=0}^{N} B_{2,k} K_{j,k}(t - h_k), \quad x(0) = B_{1,j} \qquad (3.110)$$

$$\min_{K_j} \int_0^\infty \|C_1 x(t) + \sum_{k=0}^{N} D_{2,k} K_{j,k}(t - h_k)\|_2^2 dt, \qquad (3.111)$$

where M is the number of column of B_1. Here $K_j(t)$ and $B_{1,j}$ are the respective j-th column of $K(t)$ and B_1, while $K_{j,k}(t)$, $B_{2,k}$, and $D_{2,k}$ are the respective part of $K_j(t)$, B_2, and D_2 that corresponds to the delay h_k. In this case, we have that

$$K = \begin{bmatrix} K_{j=1,k=0} & \cdots & K_{j=M,k=0} \\ \vdots & & \vdots \\ K_{j=1,k=N} & \cdots & K_{j=M,k=N} \end{bmatrix}. \qquad (3.112)$$

Following the algorithm of Theorem 3.13, first we define a new set of inputs that corresponds to the columns of $K(t)$:

$$\phi_{j,k}(t) = \begin{bmatrix} K_{j,0}(t) \\ K_{j,1}(t-h_1) \\ \vdots \\ K_{j,k}(t-h_k) \end{bmatrix} [\mathbb{1}(t-h_{k+1}) - \mathbb{1}(t-h_k)], \quad k = 0, \ldots, N-1,$$

(3.113)

$$\phi_{j,N}(t) = \begin{bmatrix} K_{j,0}(t) \\ K_{j,1}(t-h_1) \\ \vdots \\ K_{j,N}(t-h_N) \end{bmatrix} \mathbb{1}(t-h_N).$$

(3.114)

By computing the optimal $\phi_{j,k}(t)$ for $j = 1, \cdots, M$ and $k = 0, \cdots, N$, we may recover the optimal $K(t)$ using (3.71).

For a given value of j, i.e. for each column of the controller K, by applying Theorem 3.13 we may obtain the optimal $\phi_{j,k}(t)$. For $k = 1, \ldots, N-1$ it is the impulse response of the transfer function (3.95):

$$\phi_{j,k,\text{opt}}(s) = e^{-sh_k} \tau_{L_k} \left(F_k(sI - S_k)^{-1} P_k x_{j,k} \right),$$

(3.115)

where $L_k = h_{k+1} - h_k$ and the formulas for S_k, P_k, and F_k are given in Lemma 3.12. For $k = N$, $\Phi_{j,N,\text{opt}}$ is the impulse response of the transfer function (3.80):

$$\Phi_{N,j,\text{opt}}(s) = e^{-sh_N} \left(F_N(sI - S_N)^{-1} x_{j,N} \right).$$

(3.116)

The formulas for S_N and F_N are given in Lemma 3.11.

Note that the optimal state at $t = h_k$, denoted by $x_{j,k}, k = 0, \ldots, N$, is the only thing that changes in (3.115,3.116) as j changes. The values of L_k, S_k, P_k, F_k, S_N, and F_N remain the same as j varies. Moreover, the optimal state $x_{j,k}$ may be computed iteratively using (3.98):

$$x_{j,k+1} = \Gamma_k(L_k) x_{j,k},$$

(3.117)

with the function $\Gamma_k(t)$ given by (3.90). Since $x_{j,k=0}$ is equal to the j-th column of B_1, we have that

$$x_{j,k} = \left(\prod_{i=0}^{k-1} \Gamma_i(L_i) \right) B_{1,j}.$$

(3.118)

Therefore we may write for $k = 0, \ldots, N - 1$

$$
\begin{aligned}
\Phi_{k,\mathrm{opt}}(s) &= \begin{bmatrix} \phi_{1,k,\mathrm{opt}}(s) & \cdots & \phi_{M,k,\mathrm{opt}} \end{bmatrix} \\
&= e^{-sh_k} \tau_{L_k} \left(F_k(sI - S_k)^{-1} P_k \begin{bmatrix} x_{1,k} & \cdots & x_{M,k} \end{bmatrix} \right) \\
&= e^{-sh_k} \underbrace{\tau_{L_k} \left(F_k(sI - S_k)^{-1} P_k \left(\prod_{i=0}^{k-1} \Gamma_i(L_i) \right) B_1 \right)}_{=: \tilde{\Phi}_{k,\mathrm{opt}}(s)} \\
&= e^{-sh_k} \tilde{\Phi}_{k,\mathrm{opt}}(s).
\end{aligned}
\tag{3.119}
$$

Similarly, for $k = N$ we have that

$$
\begin{aligned}
\Phi_{N,\mathrm{opt}}(s) &= \begin{bmatrix} \phi_{1,N,\mathrm{opt}}(s) & \cdots & \phi_{M,N,\mathrm{opt}} \end{bmatrix} \\
&= e^{-sh_N} \left(F_N(sI - S_N)^{-1} \begin{bmatrix} x_{1,N} & \cdots & x_{M,N} \end{bmatrix} \right) \\
&= e^{-sh_N} \underbrace{\left(F_N(sI - S_N)^{-1} \left(\prod_{i=0}^{N-1} \Gamma_i(L_i) \right) B_1 \right)}_{=: \tilde{\Phi}_{N,\mathrm{opt}}(s)} \\
&= e^{-sh_N} \tilde{\Phi}_{N,\mathrm{opt}}(s)
\end{aligned}
\tag{3.120}
$$

Hence, once the matrices S_k, F_k, P_k, $k = 1, ..., N - 1$, S_N, and F_N have been computed, we may compute $\Phi_{k,\mathrm{opt}}$, $k = 0, ..., N$ using (3.119) and (3.120). Finally, the optimal controller $K_{\mathrm{opt}}(s)$ may be recovered using (3.71).

It is straightforward to show that the optimal controller $K_{\mathrm{opt}}(s)$ may be written as the sum of a FIR block and a rational transfer matrix pre-multiplied by a multiple delay operator:

$$
K_{\mathrm{opt}}(s) = \tilde{\Phi}(s) + \tilde{\Lambda}(s) K_{\mathrm{r}}(s),
\tag{3.121}
$$

where the rational transfer function

$$
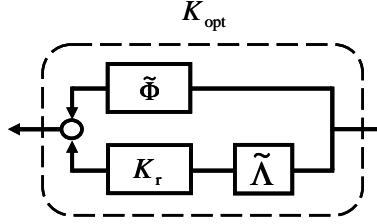K_{\mathrm{r}}(s) = \tilde{\Phi}_N(s)
\tag{3.122}
$$

Figure 3.5.: The structure of the optimal controller K_{opt} of the regulator problem.

is defined in (3.120) and

$$
\tilde{\Phi}(s) = \begin{bmatrix} \tilde{\Phi}^0_{0,\text{opt}}(s) \\ 0_{1,M} \\ \vdots \\ 0_{N-1,M} \\ 0_{N,M} \end{bmatrix} + \begin{bmatrix} e^{-sh_1}\tilde{\Phi}^0_{1,\text{opt}}(s) \\ \tilde{\Phi}^1_{1,\text{opt}}(s) \\ \vdots \\ 0_{N-1,M} \\ 0_{N,M} \end{bmatrix} + \cdots + \begin{bmatrix} e^{-s(h_{N-1})}\tilde{\Phi}^0_{N-1,\text{opt}}(s) \\ e^{-s(h_{N-1}-h_1)}\tilde{\Phi}^1_{N-1,\text{opt}}(s) \\ \vdots \\ \tilde{\Phi}^{N-1}_{N-1,\text{opt}}(s) \\ 0_{N,M} \end{bmatrix},
$$

$$
\tag{3.123}
$$

$$
\tilde{\Lambda}(s) = \begin{bmatrix} e^{-sh_N}I_0 & & & \\ & e^{-s(h_N-h_1)}I_1 & & \\ & & \ddots & \\ & & & I_N \end{bmatrix}.
\tag{3.124}
$$

Here $\tilde{\Phi}_k$, with $k = 1, \ldots, N - 1$, is defined in (3.119) and $\tilde{\Phi}^i_k$, with $k = 1, \ldots, N - 1$ and $i = 0, \ldots, k$, are the rows of $\tilde{\Phi}_k$ corresponding to the delay h_k in the delay operator (3.48). The structure of the controller is depicted in Figure 3.5. Note that all blocks in the controller of Figure 3.5 are stable, thus the controller is stable.

The state dimension of the rational part of the optimal controller is the same as the state dimension of the plant. For the case where the controller is single-column, the optimal H_2-norm of the overall transfer function $T_1(s) + T_2(s)\Lambda(s)K(s)$ is given by (3.109). For the general case where $K(s)$ has multiple columns, the optimal H_2-norm may be obtained from the following expression:

$$
\min_K \|T_1(s) + T_2(s)\Lambda(s)K(s)\|^2 = \text{trace } B_1^T M_0 B_1.
\tag{3.125}
$$

3.7. Constructing the optimal controller for the standard problem

The structure of the optimal controller $K_{\text{opt}}(s)$ of the regulator problem is discussed in the previous section. In this section, we construct the controller for the standard problem $K_{\text{s,opt}}(s)$.

The standard problem is solved by reducing it to a regulator problem using either Lemma 3.5 for the input delay case or Lemma 3.6 for the output delay case. Once the optimal controller of the regulator problem is obtained using Lemma 3.9 and Theorem 3.13, the controller for the standard problems of Figure 3.1 may be obtained from the equations (3.25,3.26) for the input delay case and from (3.39,3.40) for the output delay case. In either case the controller structure is depicted in Figure 3.6. The block $V_\bullet$ should be replaced by either by V_u (3.30) for the input delay case or by V_y (3.44) for the output delay case. The block K_{opt} is either the optimal solution of the H_2 regulator problem (3.24) for the input delay case or the optimal solution of the H_2 filtering problem (3.38) for the output delay case. The block Φ_{22} is either given by (3.27) for the input delays case or by (3.41) for the output delays case.

The state space realizations of $V_u(s)$ and $V_y(s)$ may be obtained from the left and right chain scattering representations of $G(s)$ (3.15). It may be shown that

$$
V_u(s) = \begin{bmatrix} 0 & I \\ G_{21}^{-1}(s) & -G_{21}^{-1}(s)\tilde{G}_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P + LC_{P2} & -L & -\tilde{B}_{P2} \\ \hline 0 & 0 & I \\ -R_2^{-\frac{1}{2}}C_{P2} & R_2^{-\frac{1}{2}} & 0 \end{array} \right]
$$
(3.126)

and

$$
V_y(s) = \begin{bmatrix} 0 & G_{12}^{-1}(s) \\ I & -\tilde{G}_{22}G_{12}^{-1}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P + B_{P2}F & 0 & B_{P2}R_1^{-\frac{1}{2}} \\ \hline F & 0 & R_1^{-\frac{1}{2}} \\ -\tilde{C}_{P2} & I & 0 \end{array} \right].
$$
(3.127)

Here, the matrices F, L, $\tilde{B}_{P2}$, and $\tilde{C}_{P2}$ are given by (3.8), (3.9), (3.29), and (3.43), respectively. Notice that both $V_y(s)$ and $V_u(s)$ are stable. Hence, since $\Phi_{22}(s)$ and $K_{\text{opt}}(s)$ are also stable, all blocks that form the optimal controller $K_{\text{s,opt}}(s)$ are stable.
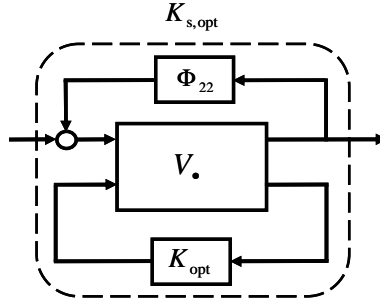
Figure 3.6.: The structure of the optimal controller $K_{\text{s,opt}}$ of the standard problem.

3.8. Numerical example

Consider the regulator problem

$$\min_{K} \|T_1(s) + T_2(s)\Lambda(s)K(s)\|_2, \tag{3.128}$$

$$\begin{bmatrix} T_1(s) & T_2(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_2 \end{array} \right] = \left[\begin{array}{c|c|cc} 0 & 1 & \begin{bmatrix} 1 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} \end{array} \right],$$

$$\Lambda(s) = \text{diag}(1, e^{-sh_1}), \quad h_1 > 0.$$

In this example $K(t)$ has a single column and thus the regulator problem may be converted directly to an LQR problem with input delays. In time domain the regulator problem (3.128) becomes

$$\min_{K} \|T_1(t) + T_2(t) * \Lambda(t) * K(t)\|_2. \tag{3.129}$$

By partitioning T_2 and K according to the delay:

$$T_2(s) = \begin{bmatrix} T_{2,0}(s) & T_{2,1}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_{2,0} & B_{2,1} \\ \hline C_1 & D_{2,0} & D_{2,1} \end{array} \right] \tag{3.130}$$

$$K(s) = \text{col}\,[K_0(s), K_1(s)], \tag{3.131}$$

where $B_{2,0} = B_{2,1} = 1$, $D_{2,0} = (0,1,0)^T$, and $D_{2,1} = (0,0,1/\alpha)^T$, the quantity inside the norm-bracket in (3.129) may be expressed as the output z of the state-equation

$$\dot{x}(t) = K_0(t) + K_1(t - h_1), \quad x(0) = B_1 = 1, \tag{3.132}$$

$$z(t) = \text{col}\,[x(t),\, K_0(t),\, \frac{1}{\alpha}K_1(t)]. \tag{3.133}$$

In this framework, the objective (3.129) becomes:

$$\min_{K_0, K_1} J(x(0), K_0(t), K_1(t)) =$$

$$\min_{K_0, K_1} \int_0^\infty \left(x(t)^2 + K_0(t)^2 + \frac{1}{\alpha^2} K_1(t - h_1)^2 \right) dt \qquad (3.134)$$

The equations (3.132,3.134) constitute the equivalent LQR problem, for which we define the new regional inputs:

$$\Phi_0(t) = K_0(t)[\mathbb{1}(t) - \mathbb{1}(t - h_1)], \qquad (3.135)$$

$$\Phi_1(t) = \text{col}\,[K_0(t),\, K_1(t - h_1)]\,\mathbb{1}(t - h_1), \qquad (3.136)$$

for the time regions $t \in [0, h_1]$ and $t \in [h_1, \infty]$, respectively. Now we may solve the problem backward in time using the algorithm of Theorem 3.13. For the time region $t \in [h_1, \infty]$, the state-space equation (3.132) becomes

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \Phi_1(t), \quad x(h_1) = x_{\text{opt}}(h_1) \qquad (3.137)$$

for a certain but not yet known $x_{\text{opt}}(h_1)$ and the cost contribution over this period is

$$J_{[h_1, \infty]} = \int_{h_1}^\infty \left(x(t)^2 + \Phi_1(t)^T \Phi_1(t) \right) dt. \qquad (3.138)$$

Using Lemma 3.12, it may be shown that the optimal input of this region is given by the state feedback

$$\Phi_{1, \text{opt}}(t) = \frac{-1}{\sqrt{1 + \alpha^2}} \begin{bmatrix} 1 \\ \alpha^2 \end{bmatrix} x(t)\,\mathbb{1}(t - h_1) \qquad (3.139)$$

resulting in the optimal state trajectory

$$x_{\text{opt}}(t) = e^{-\sqrt{1+\alpha^2}(t - h_1)} x_{\text{opt}}(h_1), \quad t \in [h_1, \infty]. \qquad (3.140)$$

Substituting (3.140) back into (3.139) we obtain the optimal input signal for $t \in [h_1, \infty]$:
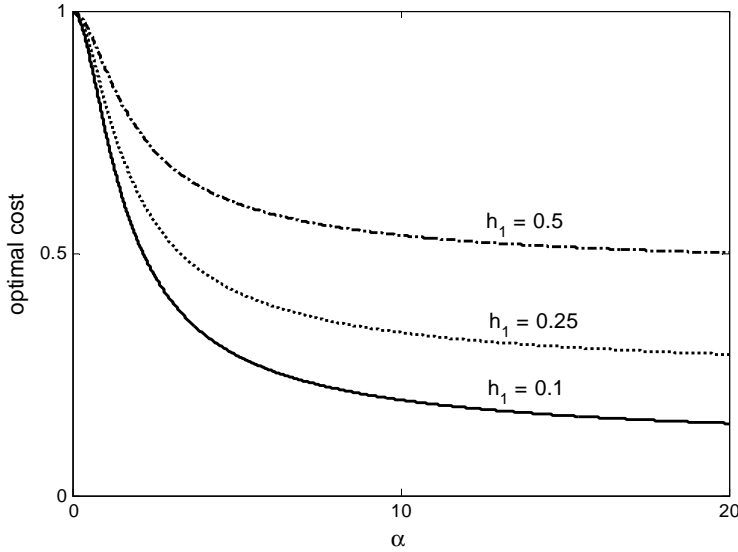
$$\Phi_{1, \text{opt}}(t) = \frac{-1}{\sqrt{1 + \alpha^2}} \begin{bmatrix} 1 \\ \alpha^2 \end{bmatrix} e^{-\sqrt{1+\alpha^2}(t - h_1)} x_{\text{opt}}(h_1)\,\mathbb{1}(t - h_1). \qquad (3.141)$$

Furthermore, the optimal cost over this region is

$$J_{[h_1, \infty], \text{opt}} = x_{\text{opt}}(h_1)^2 / (\sqrt{1 + \alpha^2}). \qquad (3.142)$$

Applying the principle of optimality, we move to the time region $[0, h_1]$, in which the state equation is given by

$$\dot{x}(t) = \Phi_0(t), \quad x(0) = 1. \qquad (3.143)$$

Figure 3.7.: The optimal cost as a function of α

The infinite horizon criterion function (3.134) may be replaced by the cost contribution over the region $[0, h_1]$ plus the quadratic final state penalty:
(3.142)

$$\int_0^{h_1} \left(x(t)^2 + \Phi_0(t)^2 \right) dt + x(h_1)^2/(\sqrt{1+\alpha^2}). \tag{3.144}$$

By solving the finite horizon LQR problem (3.143,3.144), it may be shown that the optimal input for the region $t \in [0, h_1]$ is given by:

$$\Phi_{0,\mathrm{opt}}(t) = \left[\sinh(t) - q(h_1) \cosh(t) \right] \left[\mathbb{1}(t) - \mathbb{1}(t - h_1) \right] \tag{3.145}$$

where the function $q(h_1)$ is

$$q(h_1) = \frac{\sinh(h_1) + \frac{1}{\sqrt{1+\alpha^2}} \cosh(h_1)}{\cosh(h_1) + \frac{1}{\sqrt{1+\alpha^2}} \sinh(h_1)}, \tag{3.146}$$

resulting in the optimal state trajectory

$$x_{\mathrm{opt}}(t) = \left[\cosh(t) - q(h_1) \sinh(t) \right], \quad t \in [0, h_1]. \tag{3.147}$$

We may then compute the optimal state at $t = h_1$ and the optimal cost over $t \in [0, \infty]$:

$$x_{\mathrm{opt}}(h_1) = \frac{1}{\cosh(h_1) + \frac{\sinh(h_1)}{\sqrt{1+\alpha^2}}}, \quad J_{[0,\infty],\mathrm{opt}} = q(h_1).$$

The optimal state trajectory is obtained by combining the optimal state trajectories of the two time regions (3.140,3.147). The optimal cost as a function of the parameter α for different values of the delay h_1 is shown in Figure 3.7. As expected, larger values of the delay h_1 correspond to higher cost. It is also evident that the cost decreases as the parameter α increases. This observation may be explained as follows. From the criterion function (3.134), it is obvious that larger α makes the second input $K_1(t)$ cheaper, allowing the controller to inject large input while keeping the cost relatively small. Indeed, we can see from the optimal state trajectory depicted in Figure 3.8 that for large α, as soon as the input $K_1(t)$ is active at $t = h_1$, the state is quickly driven to zero. The larger the parameter α, the quicker the state vanishes after $t = h_1$. By combining (3.141) and (3.145), we may retrieve the original input $K(t) = \mathrm{col}(K_0(t), K_1(t))$ of the LQR problem (3.132,3.134):

$$K_0(t) = \begin{cases} \sinh(t) - q(h_1)\cosh(t), & 0 \le t \le h_1 \\ \frac{-1}{\sqrt{1+\alpha^2}} x_{\mathrm{opt}}(h_1) e^{-\sqrt{1+\alpha^2}(t-h_1)}, & t > h_1 \end{cases}$$

$$K_1(t) = (-\alpha^2/(\sqrt{1+\alpha^2})) x_{\mathrm{opt}}(h_1) e^{-\sqrt{1+\alpha^2}t} \mathbb{1}(t).$$

Using formulas from Section 3.6, we may obtain the optimal controller for the regulator problem (3.128):

$$K_{\mathrm{opt}}(s) = \tau_{h_1}\left(\begin{bmatrix} \frac{-q(h_1)}{s^2-1} \\ 0 \end{bmatrix}\right) + \begin{bmatrix} e^{-sh_1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-x_{\mathrm{opt}}(h_1)}{\sqrt{1+\alpha^2}(s+\sqrt{1+\alpha^2})} \\ \frac{-\alpha^2 x_{\mathrm{opt}}(h_1)}{\sqrt{1+\alpha^2}(s+\sqrt{1+\alpha^2})} \end{bmatrix}.$$

3.9. Concluding remarks

In this chapter, a time domain solution to the standard H_2 problem with i/o delays is given. However, this approach appears only works for the cases where the delays occur on only one side of the controller. The two sided case will be solved in the next chapter, where a frequency domain approach is employed.

The controller structure resulting from the method in this chapter is quite simple. The controller consists of rational blocks, finite impulse response blocks, and delay components, all of which are implementable.

It should also be noted that the solution to the H_2-optimal control problem considered in this chapter is not only interesting from a theoretical point of view, but also has the potential for application. For example, the paper [GH98] considers the problem of steel-sheet profile control at a steel rolling mill, which has different delays in its measurement channels. The problem is formulated as an LQG problem, which is equivalent to the H_2
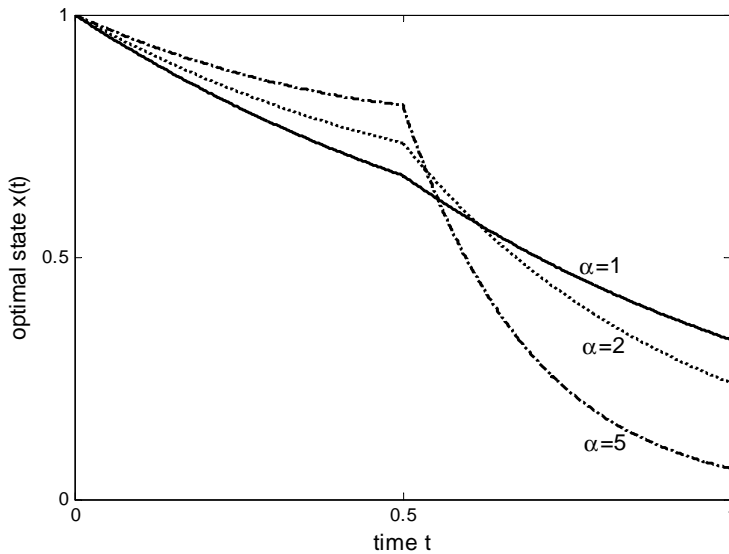
Figure 3.8.: The optimal state trajectory of the equivalent LQR problem for $h_1 = 0.5$

problem. However, the method developed in [GH98] requires a simplifying assumption that the delay operator commutes with the plant. Certain approximation has to be made to meet this assumption, resulting in a controller that is not only non-optimal but also of high order. Using the method derived in this chapter, it is possible to compute the true optimal H_2-controller for the same control problem without the approximation. This case study will be discussed in detail in Chapter 6.

4

The Standard H_2 Problem: Frequency Domain Approach

In Chapter 3, a time domain solution of the standard H_2 problem is developed. The solution is obtained via conversion to a regulator problem, which when viewed in time domain may be interpreted as an LQR problem with multiple input delays. The solution provides straightforward formulas for the optimal controller. In addition, the resulting controller structure is simple and implementable. However, the theory appears to work only for the cases where delays occur only on one side of the controller.

In this chapter, a frequency domain solution that covers the general case, where delays occur in both sides of the controller, is proposed. As in the time domain solution, the approach begins with converting the standard problem of Figure 4.1 to the simplified problem that has an advantageous feature of a less demanding stability requirement. The simplified problem only requires the stability of the transfer function from the external input to the external output compared to the internal stability requirement in the original standard problem. The conversion is done using results borrowed from [Mir03a]. It is then further transformed to what is called the two-sided regulator problem of Figure 4.2. The latter is solved using orthogonal projection arguments and spectral factorization. Similar to the time domain solution, the resulting optimal controller consists of rational blocks, finite impulse response (FIR) blocks, and delay operators, all of which are implementable. A drawback of the method developed in this chapter is that it cannot handle unstable plants. This chapter is based on the papers [MMK03] and [MMK05].

4.1. The standard H_2 problem with two sided delays

We consider standard control systems in which time delays are present in the control input and the measurement output. Such control systems are
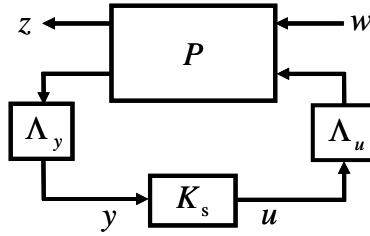
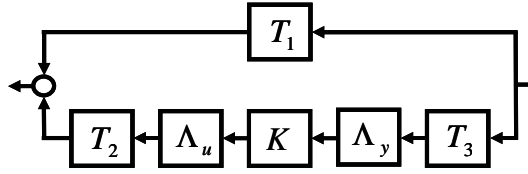Figure 4.1.: Standard control system with i/o delays.

Figure 4.2.: Two sided regulator problem.

depicted in Figure 4.1. Here the plant P is a rational transfer matrix which is assumed of having the following realization

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & B_{P1} & B_{P2} \\ \hline C_{P1} & 0 & D_{P12} \\ C_{P2} & D_{P21} & 0 \end{array} \right] \tag{4.1}$$

interconnected with a proper controller K_s and the multiple delay operators of the form

$$\Lambda_y(s) = \text{diag}(e^{-sh_{y1}}, e^{-sh_{y2}}, ..., e^{-sh_{ym}}), \tag{4.2}$$
$$\Lambda_u(s) = \text{diag}(e^{-sh_{u1}}, e^{-sh_{u2}}, ..., e^{-sh_{up}}), \tag{4.3}$$

where m and p are the dimension of y and u, respectively. We assume that P satisfies the following standard assumptions:

A1 (C_{P2}, A_P, B_{P2}) is detectable and stabilizable;

A2 $R_1 = D_{P12}^T D_{P12} > 0$ and $R_2 = D_{P21} D_{P21}^T > 0$,

A3 $\begin{bmatrix} A_P - j\omega I & B_{P2} \\ C_{P1} & D_{P12} \end{bmatrix}$ and $\begin{bmatrix} A_P - j\omega I & B_{P1} \\ C_{P2} & D_{P21} \end{bmatrix}$ have full column rank and full row rank, respectively $\forall \omega \in \mathbb{R}$.

As in the previous chapter, the upper left block of the D-matrix of P, denote by D_{P11}, is assumed to be zero. Without loss of generality, the lower right block, denoted by D_{P22}, is also assumed to be zero. If D_{P22} is nonzero, we may first find the controller for the case where it is zero and later recover the controller for the nonzero case by connecting $\Lambda_y(s)D_{P22}\Lambda_u(s)$ as a negative feedback to the controller. See Section 2.2 for details.

Problem 4.1 (Standard H_2 problem with two sided delays). Consider the control systems of Figure 4.1 where the plant $P(s)$ and the delay operators $\Lambda_u(s)$ and $\Lambda_y(s)$ are given by (4.1,4.2,4.3). Suppose that the assumptions A1, A2, and A3 on page 64 are satisfied. Find a stabilizing LTI causal controller $K_s(s)$ such that the H_2-norm of the transfer function from w to z is minimized.

4.2. Conversion to two-sided regulator problem

The aim is to transform the standard problem corresponding to Figure 4.1 of minimizing

$$\|F_\ell(P(s), \Lambda_u(s)K_s(s)\Lambda_y(s))\|_2$$

to the two sided regulator problem corresponding to Figure 4.2 of minimizing

$$\|T_1(s) + T_2(s)\Lambda_u(s)K(s)\Lambda_y(s)T_3(s)\|_2,$$

where there is a proper bijection between K_s and K.

The transformation is carried out in two stages. The first stage is identical to the first stage of the conversion in Section 3.3, which is formulated in Lemma 3.3. There, the standard problem is converted to the simplified problem, in which the internal stability requirement of the original problem is transformed to a less demanding condition of the stability of the transfer function from the external input to the external output. This allows the second stage, which further transforms the problem to the two-sided regulator problem. An important feature of the latter problem is that with the help of Lemma 2.5, it may be proved that we may restrict ourself to stable controllers in the optimization. The formulas needed for the complete conversion from the standard problem to the two sided regulator problem is summarized in the following theorem.

Theorem 4.2 (From standard to two sided regulator problem). *Consider Problem 4.1. Let X and Y be the stabilizing solutions of the following Riccati equations,*

$$A_X^T X + X A_X - X B_{P2} R_1^{-1} B_{P2}^T X + C_{P1}(I - D_{P12} R_1^{-1} D_{P12}^T) C_{P1} = 0, \tag{4.4}$$

$$A_Y Y + Y A_Y^T - Y C_{P2}^T R_2^{-1} C_{P2} Y + B_{P1}(I - D_{P21}^T R_2^{-1} D_{P21}) B_{P1}^T = 0, \tag{4.5}$$

where

$$A_X = A_P - B_{P2} R_1^{-1} D_{P21}^T C_{P1}, \quad A_Y = A_P - B_{P1} D_{P21}^T R_2^{-1} C_{P2}$$

$$R_1 = D_{P12}^T D_{P12} > 0, \quad R_2 = D_{P21} D_{P21}^T > 0.$$

4. The Standard H_2 Problem: Frequency Domain Approach

Define

$$F = -R_1^{-1}(B_{P2}^T X + D_{P12}^T C_{P1}), \ L = -(Y C_{P2}^T + B_{P1} D_{P21}^T) R_2^{-1}. \quad (4.6)$$

Also define the transfer function $G(s)$

$$G = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & -L R_2^{\frac{1}{2}} & B_{P2} \\ \hline -R_1^{\frac{1}{2}} F & 0 & R_1^{\frac{1}{2}} \\ C_{P2} & R_2^{\frac{1}{2}} & 0 \end{array} \right]. \quad (4.7)$$

Consider the problem of minimizing $\|F_\ell(P(s), \Lambda_u(s) K_s(s) \Lambda_y(s))\|_2$ over stabilizing, proper K_s, where $P(s)$, $\Lambda_u(s)$, and $\Lambda_y(s)$ are given by (4.1,4.2,4.3). Define K such that

$$K(s) = K_s(s) \left(I - \Lambda_y(s) G_{22}(s) \Lambda_u(s) K_s(s)\right)^{-1}, \quad (4.8)$$

$$K_s(s) = (I + K(s) \Lambda_y(s) G_{22}(s) \Lambda_u(s))^{-1} K(s). \quad (4.9)$$

Then the following statements hold:

1. *There is a proper bijection between K and K_s,*

2. *$K_s(s)$ minimizes $\|F_\ell(P(s), \Lambda_u(s) K_s(s) \Lambda_y(s))\|_2$ if and only if $K(s)$ minimizes $\|G_{11}(s) + G_{12}(s) \Lambda_u(s) K(s) \Lambda_y(s) G_{21}(s)\|_2$ where $G(s)$ is given by (4.7),*

3. *$F_\ell(P(s), \Lambda_u(s) K_s(s) \Lambda_y(s))$ is internally stable if and only if*

$$(G_{11}(s) + G_{12}(s) \Lambda_u(s) K(s) \Lambda_y(s) G_{21}(s)) \in H_\infty,$$

4. *the squared optimal H_2-norm is given by*

$$\min_{K_s} \|F_\ell(P(s), \Lambda_u(s) K_s(s) \Lambda_y(s))\|_2^2 = \text{tr}(B_{P1}^T X B_{P1}) + \text{tr}(R_1 F Y F^T)$$
$$+ \min_K \|G_{11}(s) + G_{12}(s) \Lambda_u(s) K(s) \Lambda_y(s) G_{21}(s)\|_2^2,$$

$$(4.10)$$

5. *if $F_\ell(P(s), \Lambda_u(s) K_s(s) \Lambda_y(s))$ is internally stable and*

$$(G_{11}(s) + G_{12}(s) \Lambda_u(s) K(s) \Lambda_y(s) G_{21}(s)) \in H_2,$$

then $K(s) \in H_2$, implying that $K(s)$ may be restricted to transfer functions in H_2 in the minimization of $\|G_{11}(s) + G_{12}(s) \Lambda_u(s) K(s) \Lambda_y(s) G_{21}(s)\|_2$.

Proof. Statement 1 follows from the fact that $G_{22}(s)$ strictly proper. Suppose $K(s)$ is proper, then there exist a $\rho \in \mathbb{R}$ such that

$$\lim_{s \to \infty, \text{Re}(s) > \rho}(I + K(s)\Lambda_y(s)G_{22}(s)\Lambda_u(s))^{-1} = I$$

implying that $K_s(s) = (I + K(s)\Lambda_y(s)G_{22}(s)\Lambda_u(s))^{-1}K(s)$ is also proper. The converse may be proved similarly.

To prove statement 2, recall that $K_s(s)$ minimizes

$$\|F_\ell(P(s), \Lambda_u(s)K_s(s)\Lambda_y(s))\|_2$$

if and only if $K_s(s)$ minimizes $\|F_\ell(G(s), \Lambda_u(s)K_s(s)\Lambda_y(s))\|_2$ with $G(s)$ given by (4.7) (Lemma 3.3, Statement 1). Furthermore, we have that

$$\begin{aligned}
F_\ell(G(s), \Lambda_u(s)K_s(s)\Lambda_y(s)) &= F_\ell(\begin{bmatrix} G_{11}(s) & G_{12}(s)\Lambda_u(s) \\ \Lambda_y(s)G_{21}(s) & \Lambda_y(s)G_{22}(s)\Lambda_u(s) \end{bmatrix}, K_s(s)) \\
&= F_\ell(\begin{bmatrix} G_{11}(s) & G_{12}(s)\Lambda_u(s) \\ \Lambda_y(s)G_{21}(s) & 0 \end{bmatrix}, K(s)) \\
&= G_{11}(s) + G_{12}(s)\Lambda_u(s)K(s)\Lambda_y(s)G_{21}(s).
\end{aligned}$$

$$(4.11)$$

Statements 3 and 4 follow from statements 2 and 3 of Theorem 3.2. What is left is proving statement 5. First note that $K(s)$ may be written as

$$K = \tilde{K}_s(I - \tilde{G}_{22}\tilde{K}_s)^{-1}, \tag{4.12}$$

where

$$\tilde{G}_{22}(s) = \tilde{C}_{P2}(sI - A_P)^{-1}\tilde{B}_{P2}, \tag{4.13}$$

$$\tilde{K}_s(s) = (I + K_s\Phi_{22})^{-1}K_s, \tag{4.14}$$

$$\Phi_{22}(s) = \tilde{G}_{22}(s) - \Lambda_y(s)G_{22}(s)\Lambda_u(s), \tag{4.15}$$

with $\tilde{C}_{P2}$ and $\tilde{B}_{P2}$ given by (2.27,2.28). Next, recall that a controller internally stabilizes a rational plant $P(s)$ if and only if it stabilizes the plant's (2,2) part (see e.g. Lemma 11.2 of [ZD98]). It follows from Lemma 2.5, that it is also true even if delays are present in the i/o channels. Now, suppose the closed loop system associated with $F_\ell(P(s), \Lambda_u(s)K_s(s)\Lambda_y(s))$ is internally stable. Then since $P_{22}(s) = G_{22}(s)$, $K_s(s)$ also stabilizes $\Lambda_y(s)G_{22}(s)\Lambda_u(s)$. It follows from Lemma 2.5 that $\tilde{K}_s(s)$ stabilizes $\tilde{G}_{22}(s)$, implying that the closed loop system associated with

$$F_\ell(\begin{bmatrix} 0 & I \\ I & \tilde{G}_{22}(s) \end{bmatrix}, \tilde{K}_s(s))$$

is internally stable. Hence, we have that

$$F_\ell(\begin{bmatrix} 0 & I \\ I & \tilde{G}_{22}(s) \end{bmatrix}, \tilde{K}_s(s)) = K(s) \in H_\infty.$$

Now suppose that $H(s) = (G_{11}(s) + G_{12}(s)\Lambda_u(s)K(s)\Lambda_y(s)G_{21}(s)) \in H_2$. Define $K_\Lambda(s) = \Lambda_u(s)K(s)\Lambda_y(s) \in H_\infty$. It is clear that if $K(s) \in H_\infty$ and $K_\Lambda(s) \in H_2$, then $K(s) \in H_2$.

Thus we only need to prove that $K_\Lambda(s) \in H_2$. For this, first define a disc on the complex plane of radius R centered at the origin with R sufficiently large such that all poles of $G_{12}^{-1}(s)$, $G_{21}^{-1}(s)$, and $G_{11}(s)$ are well inside the disc. The area inside the disc, i.e. $|s| \leq R$ is denoted by A_{in}, while the area outside the disc, i.e. $|s| > R$ is denoted by A_{out}.

To ascertain whether or not $K_\Lambda(s)$ is in H_2, we compute:

$$\sup_{\sigma>0} \left\{ \int_{-\infty}^{\infty} \text{trace}[K_\Lambda^*(\sigma + j\omega)K_\Lambda(\sigma + j\omega)]d\omega \right\}. \tag{4.16}$$

We can split the integration in (4.16) into two parts: the part corresponding to the area inside the disc (A_{in}) plus the part corresponding to the area outside the disc (A_{out}).

Since $K_\Lambda(s) \in H_\infty$, for $\sigma > 0$ the part of (4.16) corresponding to A_{in} is bounded from above by $2nR\|K_\Lambda(s)\|_\infty^2 < \infty$, where n is a positive integer that depends on the dimension of K_Λ.

Having determined that the integration part corresponding to A_{in} is finite with a bound that is independent of σ, we evaluate the part corresponding to A_{out}. First define

$$M := \max_{|s|>R, \text{Re}(s)>0} (\bar{\sigma}(G_{12}^{-1}(s), \bar{\sigma}(G_{21}^{-1}(s)) < \infty.$$

Next, recall that we can write $K_\Lambda(s) = G_{12}^{-1}(s)(H(s) - G_{11}(s))G_{21}^{-1}(s)$, so that

$$\int_{A_{\text{out}}, \text{Re}(s)>0} \text{trace}[K_\Lambda^*(\sigma + j\omega)K_\Lambda(\sigma + j\omega)]d\omega$$
$$\leq M^4 \sup_{\sigma>0} \int_{A_{\text{out}}} \|(H(\sigma + j\omega) - G_{11}(\sigma + j\omega))\|_2^2 d\omega. \tag{4.17}$$

The right hand side of (4.17) is finite, with a bound that is independent of σ. This follows from the fact that $G_{11}(s)$ is rational and strictly proper with no poles in A_{out} and that $H(s)$ is in H_2. Hence, $K_\Lambda(s) \in H_2$, implying that $K(s) \in H_2$.

$\square$

4.3. Two-sided regulator problem

In the previous section, it is shown that the H_2-control problem corresponding to the standard configuration of Figure 2.1 is equivalent to the two-sided regulator problem of Figure 4.2. In this section and the subsequent sections, the latter problem is solved.

First let us formulate the two sided regulator problem more precisely.

Problem 4.3 (Two sided H_2 regulator problem). The two-sided H_2 regulator problem is finding a stable LTI controller K that minimizes

$$\|T_1(s) + T_2(s)\Lambda_u(s)K(s)\Lambda_y(s)T_3(s)\|_2 \tag{4.18}$$

where $T_1(s)$, $T_2(s)$, and $T_3(s)$ are rational transfer functions that satisfy the following conditions:
A4 $T_1(s) \in H_2$ and $T_2(s), T_3(s) \in H_\infty$;
A5 $T_2(s)$ and $T_3(s)$ have respectively full column rank and full row rank on $j\mathbb{R} \cup \infty$,

The delay operators Λ_u and Λ_y are given by (4.2,4.3). Without loss of generality, it is assumed that the delays in the delay operators are ordered according to their magnitude:
A6 $\Lambda_u(s)$ and $\Lambda_y(s)$ are of the form[1]

$$\begin{aligned} \Lambda_\bullet(s) &= \mathrm{diag}\left(e^{-sh_0}I_0, e^{-sh_1}I_1, \ldots, e^{-sh_N}I_N\right), \\ 0 &= h_0 < h_1 < \cdots < h_N, \end{aligned} \tag{4.19}$$

where N is the number of nonzero distinct delays. Here, I_0 may be empty indicating that there are no channels that are not delayed.

Remark 4.4 (One sided H_2 regulator problem). A special case of the problem (4.18) where $\Lambda_y(s) = I, T_3(s) = I$ is called one-sided regulator problem. This problem, except for the stability condition imposed on T_1 and T_2 is identical to Problem 3.8 of Chapter 3.

Remark 4.5 (Stability assumption). In the formulation of the two sided regulator problem, $T_1(s), T_2(s)$, and $T_3(s)$ are assumed to be stable. This means that only stable plants can be handled by the method of this chapter, at least in its present form.

The solution of the two-sided problem (4.18) is based on the solution of its special case, the one-sided regulator problem where $\Lambda_y(s) = I, T_3(s) = I$. In the next section, the solution to the one-sided problem is discussed, followed by a section elaborating the two-sided problem solution.

---

[1]The structure of $\Lambda_u(s)$ and $\Lambda_y(s)$ is given by (4.19). However, the number N and the dimension of the identity matrices $I_0, \ldots, I_N$ may differ in the two delay operator.

4.4. Solution to the one-sided regulator problem

This section provides the solution to a special case of the two sided problem of minimizing (4.18) where $\Lambda_y(s) = I, T_3(s) = I$, i. e.

$$\min_{K \in H_2} \|T_1(s) + T_2(s)\Lambda_\bullet(s)K(s)\|_2.$$

The problem is solved by transforming the one-sided problem to another one-sided problem with one less distinct delays in the delay operator. Hence, by applying this transformation successively, we may reduce the delay operator to one with only two distinct delays. The simple structure of the delay operator in the reduced problem allows us to obtain the optimal controller.

The solution is based on spectral factorization theory, projection arguments, and a special decomposition of the delay operators. The subsequent lemmas provides the necessary ingredients for solving the one-sided regulator problem. The first lemma, which is based on the results in [MZ00], presents the spectral factorization of a rational transfer matrix that is multiplied by a delay operator containing at most two distinct delays, which is called *simple delay operator*.

Lemma 4.6 (Spectral factorization of delayed transfer functions). *Let $F(s)$ be a stable rational transfer function that has full column rank on $j\mathbb{R} \cup \infty$, and let $\Lambda_\mathrm{s}(s)$ be a delay operator with at most two distinct delays of the form*

$$\Lambda_\mathrm{s} = \begin{bmatrix} e^{-sh_1}I_1 & 0 \\ 0 & e^{-sh_2}I_2 \end{bmatrix}, \quad h_2 \geq h_1 \geq 0. \tag{4.20}$$

Define[2] $\Pi(s) = F^\sim(s)F(s)$ so that with the appropriate partitioning we have that

$$\Lambda_\mathrm{s}^\sim(s)F^\sim(s)F(s)\Lambda_\mathrm{s}(s) = \begin{bmatrix} \Pi_{11}(s) & e^{-s(h_2-h_1)}\Pi_{12}(s) \\ e^{s(h_2-h_1)}\Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix}. \tag{4.21}$$

Define $\Phi_F(s)$ and $R(s)$ such that

$$R(s) = e^{-s(h_2-h_1)}\Pi_{11}^{-1}(s)\Pi_{12}(s) + \Phi_F(s) \tag{4.22}$$

with $R(s)$ rational and $\Phi_F(s)$ stable. Also define

$$W(s) = \begin{bmatrix} \Pi_{11}(s) & \Pi_{11}(s)R(s) \\ R^\sim(s)\Pi_{11}(s) & \Pi_{22}(s) - \Pi_{21}(s)\Pi_{11}^{-1}(s)\Pi_{12}(s) + R^\sim(s)\Pi_{11}(s)R(s) \end{bmatrix}. \tag{4.23}$$

Then $W(s)$ is rational and proper. Define $\bar{F}_\mathrm{o}(s)$ such that $W(s) = \bar{F}_\mathrm{o}^\sim(s)\bar{F}_\mathrm{o}(s)$, with $\bar{F}_\mathrm{o}(s)$ bistable. Then we have that

$$\Lambda_\mathrm{s}^\sim(s)F^\sim(s)F(s)\Lambda_\mathrm{s}(s) = \begin{bmatrix} I_1 & 0 \\ -\Phi_{\widetilde{F}}^\sim(s) & I_2 \end{bmatrix} \bar{F}_\mathrm{o}^\sim(s)\bar{F}_\mathrm{o}(s) \begin{bmatrix} I_1 & -\Phi_F(s) \\ 0 & I_2 \end{bmatrix}, \tag{4.24}$$

---

[2]Here we define $F^\sim(s) := F^T(-s)$.

and finding the spectral factorization of $\Lambda_s^\sim(s)F^\sim(s)F(s)\Lambda_s(s)$ amounts to finding the spectral factorization of $W(s)$.

Proof. First note that since $F(s)$ has full column rank on $j\mathbb{R} \cup \infty$, $\Pi_{11}(s)$ is invertible. Furthermore, proposition 2.6 shows that the decomposition (4.22) always exists. Furthermore, let us define

$$W(s) = \begin{bmatrix} I_1 & 0 \\ \Phi_{\widetilde{F}}(s) & I_2 \end{bmatrix} \Lambda_s^\sim(s)F^\sim(s)F(s)\Lambda_s(s) \begin{bmatrix} I_1 & \Phi_F(s) \\ 0 & I_2 \end{bmatrix}$$

$$= \begin{bmatrix} I_1 & 0 \\ \Phi_{\widetilde{F}}(s) & I_2 \end{bmatrix} \begin{bmatrix} \Pi_{11}(s) & e^{-s(h_2-h_1)}\Pi_{12}(s) \\ e^{s(h_2-h_1)}\Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix} \begin{bmatrix} I_1 & \Phi_F(s) \\ 0 & I_2 \end{bmatrix},$$
(4.25)

which by using the decomposition (4.22) leads to the expression (4.23). Hence, finding the spectral factorization of $\Lambda_s^\sim(s)F^\sim(s)F(s)\Lambda_s(s)$ amounts to finding the spectral factorization of $W(s)$. It may be shown that $W(s)$ defined by (4.23) is indeed proper, and that there exist a bistable transfer function $\bar{F}_o(s)$ such that $W(s) = \bar{F}_o^\sim(s)\bar{F}_o(s)$. This is proved in Section 4.7. $\quad\square$

The next lemma is a well-known orthogonal projection result (see e.g. [You88]).

Lemma 4.7 (Projection lemma). *Suppose $F_1(s), K(s) \in H_2$, $F_2(s) \in H_\infty$, $F_2^\sim(s)F_2(s) = I$, and $F_2^\sim(s)F_1(s)$ is anti-stable, then the following holds:*

$$\|F_1(s) + F_2(s)K(s)\|_2^2 = \|F_1(s)\|_2^2 + \|K(s)\|_2^2.$$
(4.26)

It follows that $\|F_1(s) + F_2(s)K(s)\|_2$ is minimized over $K(s) \in H_2$ for $K(s) = 0$.

Proof.

$$\|F_1(s) + F_2(s)K(s)\|_2^2$$
$$= \langle F_1(s), F_1(s)\rangle + 2\operatorname{Re}\langle F_1(s), F_2(s)K(s)\rangle + \langle F_2(s)K(s), F_2(s)K(s)\rangle$$
$$= \|F_1(s)\|_2^2 + 2\operatorname{Re}\langle F_2^\sim(s)F_1(s), K(s)\rangle + \|K(s)\|_2^2$$
(4.27)

Since $F_2^\sim(s)F_1(s) \in H_2^\perp$, we have that $\langle F_2^\sim(s)F_1(s), K(s)\rangle = 0 \,\forall\, K(s) \in H_2$. $\quad\square$

The last ingredient that is needed before the solution to the one-sided problem may be presented is a decomposition of the delay operator. Consider the delay operator (4.19) that contains N nonzero delays. The delay operator (4.19) may be factorized to two factors: a simple delay operator containing one nonzero delay and a multiple delay operator containing

$N - 1$ nonzero delays.

$$\Lambda_\bullet(s) = \underbrace{\left[\begin{array}{cc} I_0 & 0 \\ 0 & e^{-sh_1}\operatorname{diag}(I_1, \ldots, I_N) \end{array}\right]}_{=\Lambda_\mathrm{s}(s)} \underbrace{\operatorname{diag}(I_0, I_1, \tilde{\Lambda}_\mathrm{r}(s))}_{=\Lambda_\mathrm{r}(s)} \qquad (4.28)$$

$$\text{with } \tilde{\Lambda}_\mathrm{r}(s) = \operatorname{diag}(e^{-s(h_2 - h_1)}I_2, \ldots, e^{-s(h_N - h_1)}I_N). \qquad (4.29)$$

Hence, by extracting the simple delay operator $\Lambda_\mathrm{s}(s)$ from (4.19), the remaining factor $\Lambda_\mathrm{r}(s)$ contains one less nonzero delays.

Now that all the ingredients are in place, we are ready to state the theorem that solves the one-sided regulator problem.

Theorem 4.8 (Delay reduction theorem). *Consider the regulator problem*

$$\min_{K \in H_2} \|T_1(s) + T_2(s)\Lambda_\bullet(s)K(s)\|_2 \qquad (4.30)$$

where $T_1(s)$ and $T_2(s)$ are rational transfer functions satisfying the assumptions

- *$T_1(s) \in H_2$, $T_2(s) \in H_\infty$;*

- *$T_2(s)$ has full column rank on $j\mathbb{R} \cup \infty$,*

and the delay operator $\Lambda_\bullet(s)$ is of the form (4.19) containing N nonzero delays. Suppose $\Lambda_\bullet(s)$ is factorized according to (4.28,4.29):

$$\Lambda_\bullet(s) = \Lambda_\mathrm{s}(s)\Lambda_\mathrm{r}(s) = \Lambda_\mathrm{s}(s)\operatorname{diag}(I_0, I_1, \tilde{\Lambda}_\mathrm{r}(s)),$$

where $\Lambda_\mathrm{s}(s)$ and $\Lambda_\mathrm{r}(s)$ contain one and $(N-1)$ nonzero delays, respectively. Then there exists rational transfer functions $\tilde{T}_1(s), \tilde{T}_2(s)$ and a non-rational transfer function $\Psi(s)$ having the properties[3]

- *$\tilde{T}_1(s) \in H_2$, $\tilde{T}_2(s), \tilde{T}_2^{-1}(s) \in H_\infty$;*

- *$\Psi(s), \Psi^{-1}(s) \in H_\infty$;*

- *$\Psi(s)$ has finite impulse response,*

such that

$$\min_{K \in H_2} \|T_1(s) + T_2(s)\Lambda_\bullet(s)K(s)\|_2^2$$
$$= \|T_1(s) - T_2(s)\Lambda_\mathrm{s}(s)\tilde{T}_2^{-1}(s)\tilde{T}_1(s)\|_2^2 + \min_{\tilde{K} \in H_2} \|\tilde{T}_1(s) + \tilde{T}_2(s)\Lambda_\mathrm{r}(s)\tilde{K}(s)\|_2^2$$
$$(4.31)$$

where there is a proper bijection between K and $\tilde{K}$:

$$\tilde{K}(s) = \Psi(s)K(s). \qquad (4.32)$$

---

[3]See the proof for the construction of $\tilde{T}_1(s), \tilde{T}_2(s)$ and $\Psi(s)$.

Proof. Consider the problem (4.30). We begin by applying the factorization (4.28) to obtain

$$\|T_1(s) + T_2(s)\Lambda_\bullet(s)K(s)\|_2 = \|T_1(s) + T_2(s)\Lambda_s(s)\Lambda_r(s)K(s)\|_2. \quad (4.33)$$

Thus by absorbing $\Lambda_s(s)$ into the controller, the problem (4.30) is equivalent to the problem

$$\min_{K_r \in H_2} \|T_1(s) + T_2(s)\Lambda_s(s)K_r(s)\|_2, \quad (4.34)$$

under the condition

$$K_r(s) = \Lambda_r(s)K(s), \ K(s) \in H_2. \quad (4.35)$$

For the moment, let us forget about the condition (4.35) and concentrate in solving the problem (4.34). By construction $\Lambda_s(s)$ contains two distinct delays and is of the form (4.20), while by assumption $T_2(s)$ is stable and has full column rank on $j\mathbb{R} \cup \infty$. Hence, the spectral factorization of $T_2(s)\Lambda_s(s)$ may be computed using Lemma 4.6:

$$\Lambda_s^\sim(s)T_2^\sim(s)T_2(s)\Lambda_s(s) = T_{2,o}^\sim(s)T_{2,o}(s) \quad (4.36)$$

$$\text{where} \quad T_{2,o}(s) = \bar{T}_{2,o}(s) \begin{bmatrix} I_0 & -\Phi_T(s) \\ 0 & \text{diag}(I_1, \ldots, I_N) \end{bmatrix} \quad (4.37)$$

for a certain bistable rational transfer function $\bar{T}_{2,o}(s)$ and a certain stable non-rational transfer function $\Phi_T(s)$ that may be chosen to have finite impulse response. By defining the inner[4] transfer function

$$T_{2,i}(s) = T_2(s)\Lambda_s(s)T_{2,o}^{-1}(s), \quad (4.38)$$

we may write the inner-outer factorization of $T_2(s)\Lambda_s(s)$:

$$T_2(s)\Lambda_s(s) = T_{2,i}(s)T_{2,o}(s). \quad (4.39)$$

Substituting (4.39) into the problem (4.34) we obtain

$$\min_{K_r \in H_2} \|T_1(s) + T_2(s)\Lambda_s(s)K_r(s)\|_2 = \min_{K_r \in H_2} \|T_1(s) + T_{2,i}(s)T_{2,o}(s)K_r(s)\|_2. \quad (4.40)$$

Now suppose we split $K_r(s)$ to two parts, a certain fixed part $K_{r,f}(s)$ and a variable part $K_{r,v}(s)$:

$$K_r(s) = K_{r,f}(s) + K_{r,v}(s), \quad (4.41)$$

then we may reformulate the problem (4.40) as

$$\min_{K_{r,v} \in H_2} \| \underbrace{(T_1(s) + T_{2,i}(s)T_{2,o}(s)K_{r,f}(s))}_{=:F_1(s)} + \underbrace{T_{2,i}(s)}_{=:F_2(s)} T_{2,o}(s)K_{r,v}(s)\|_2 \quad (4.42)$$

---

[4]A transfer matrix $F(s)$ is said to be inner if $F(s)$ is stable and $F^\sim(s)F(s) = I$.

provided that (4.41) is satisfied. Define F_1 and F_2 as in (4.42) and define the fixed part of $K_\mathrm{r}(s)$ as[5]

$$K_{\mathrm{r,f}}(s) := -T_{2,\mathrm{o}}^{-1}(s)\{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+ \in H_2 \tag{4.43}$$

then we have that

$$F_1(s) = T_1(s) - T_{2,\mathrm{i}}(s)\{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+ \in H_2$$

and

$$F_2^{\sim}(s)F_1(s) = (T_{2,\mathrm{i}}^{\sim}(s)T_1(s) - \{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+) \in H_2^{\perp}.$$

By Lemma 4.7, it follows that

$$\min_{K_{\mathrm{r,v}}\in H_2} \| \left(T_1(s) + T_{2,\mathrm{i}}(s)T_{2,\mathrm{o}}(s)K_{\mathrm{r,f}}(s)\right) + T_{2,\mathrm{i}}(s)T_{2,\mathrm{o}}(s)K_{\mathrm{r,v}}(s)\|_2^2$$

$$= \min_{K_{\mathrm{r,v}}\in H_2} \left(\|T_1(s) + T_{2,\mathrm{i}}(s)T_{2,\mathrm{o}}(s)K_{\mathrm{r,f}}(s)\|_2^2 + \|T_{2,\mathrm{o}}(s)K_{\mathrm{r,v}}(s)\|_2^2\right) \tag{4.44}$$

$$= \|T_1(s) + T_{2,\mathrm{i}}(s)T_{2,\mathrm{o}}(s)K_{\mathrm{r,f}}(s)\|_2^2 + \min_{K_{\mathrm{r,v}}\in H_2} \|T_{2,\mathrm{o}}(s)K_{\mathrm{r,v}}(s)\|_2^2.$$

For the special case of the problem (4.30) where the delay operator $\Lambda_\bullet(s)$ is already a simple delay operator of the form (4.20), we may choose $\Lambda_\mathrm{r}(s) = I$ so that the condition (4.35) becomes $K(s) = K_\mathrm{r}(s)$ and the problem (4.30) is solved by taking $K_{\mathrm{r,v}}(s) = 0$ in the equivalent problem (4.44). This results in the optimal controller given by $K_{\mathrm{opt}}(s) = K_{\mathrm{r,opt}}(s) = K_{\mathrm{r,f}}(s)$, where $K_{\mathrm{r,f}}(s)$ is given by (4.43)[6].

Now let us return to the general problem. In this case we cannot take $K_{\mathrm{r,v}}(s) = 0$, because it will result in a non-causal controller K. The solution is explained in what follows. From (4.44), we notice that the original problem (4.30) is equivalent to the problem

$$\min_{K_{\mathrm{r,v}}(s)\in H_2} \|T_{2,\mathrm{o}}(s)K_{\mathrm{r,v}}(s)\|_2 \tag{4.45}$$

under the conditions (4.35) and (4.41):

$$K_\mathrm{r}(s) = \Lambda_\mathrm{r}(s)K(s) = \mathrm{diag}(I_0, I_1, \tilde{\Lambda}_\mathrm{r}(s))K(s)$$

$$K_\mathrm{r}(s) = K_{\mathrm{r,f}}(s) + K_{\mathrm{r,v}}(s) = -T_{2,\mathrm{o}}^{-1}(s)\{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+ + K_{\mathrm{r,v}}(s).$$

Substituting the second condition (4.41) into the problem (4.45) we obtain

$$\|T_{2,\mathrm{o}}(s)K_{\mathrm{r,v}}(s)\|_2 = \| - T_{2,\mathrm{o}}(s)K_{\mathrm{r,f}}(s) + T_{2,\mathrm{o}}(s)K_\mathrm{r}(s)\|_2$$
$$= \|\{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+ + T_{2,\mathrm{o}}(s)K_\mathrm{r}(s)\|_2. \tag{4.46}$$

---

[5]The operation $\{F(s)\}_+$ denotes the stable part $F(s)$.
[6]This corollary is formally stated later in Corollary 4.9.

We proceed with substituting the first condition (4.35) and the expression (4.37) for $T_{2,\mathrm{o}}(s)$ into (4.46):

$$\|\{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+ + T_{2,\mathrm{o}}(s)K_\mathrm{r}(s)\|_2$$

$$= \|\{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+ + \bar{T}_{2,\mathrm{o}}(s)\begin{bmatrix} I_0 & -\Phi_T(s) \\ 0 & \mathrm{diag}(I_1, I_r) \end{bmatrix} \mathrm{diag}(I_0, I_1, \tilde{\Lambda}_\mathrm{r}(s))K(s)\|_2,$$

where $I_r = \mathrm{diag}(I_2, \ldots, I_N)$, and by appropriately partitioning the transfer function $\Phi_T(s)$ to $[\Phi_{T1}v \ \Phi_{T2}(s)]$,

$$= \|\{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+ + \bar{T}_{2,\mathrm{o}}(s)\begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & \tilde{\Lambda}_\mathrm{r}(s) \end{bmatrix}\begin{bmatrix} I_0 & -\Phi_{T1}(s) & -\Phi_{T2}(s)\tilde{\Lambda}_\mathrm{r}(s) \\ 0 & I_1 & 0 \\ 0 & 0 & I_r \end{bmatrix} K(s)\|_2. \tag{4.47}$$

Hence, we obtain the equivalent problem (4.31) by setting

$$\tilde{T}_1(s) = \{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+, \tag{4.48}$$

$$\tilde{T}_2(s) = \bar{T}_{2,\mathrm{o}}(s), \tag{4.49}$$

$$\Psi = \begin{bmatrix} I_0 & -\Phi_{T1}(s) & -\Phi_{T2}(s)\tilde{\Lambda}_\mathrm{r}(s) \\ 0 & I_1 & 0 \\ 0 & 0 & I_r \end{bmatrix} \tag{4.50}$$

in (4.44) and (4.47). Notice that $\tilde{T}_1(s)$ is stable, $\tilde{T}_2(s)$ is bistable and rational, and $\Psi(s)$ is bistable and has a realization with finite impulse response. What remains is to show that $\tilde{T}_1(s)$ is rational. First note that according to Lemma 4.6, $\Phi_T(s)$ is given by

$$\Phi_T(s) = R(s) - e^{-sh_1}\Pi_{11}^{-1}(s)\Pi_{12}(s),$$

for certain rational transfer matrices R, $\Pi_{12}(s)$, and $\Pi_{11}(s)$. The fact that $\tilde{T}_1(s)$ is rational may be observed from the following expressions.

$$\tilde{T}_1(s) = \{T_{2,\mathrm{i}}^{\sim}(s)T_1(s)\}_+ = \{(T_2(s)\Lambda_\mathrm{s}(s)T_{2,\mathrm{o}}^{-1}(s))^{\sim}T_1(s)\}_+$$

$$= \left\{\left(T_2(s)\begin{bmatrix} I_0 & 0 \\ 0 & e^{-sh_1}I_{1\sim N} \end{bmatrix}\begin{bmatrix} I_0 & \Phi_T(s) \\ 0 & I_{1\sim N} \end{bmatrix}\bar{T}_{2,\mathrm{o}}^{-1}(s)\right)^{\sim} T_1(s)\right\}_+$$

$$= \left\{\left(T_2\begin{bmatrix} I_0 & R(s) - e^{-sh_1}\Pi_{11}^{-1}(s)\Pi_{12}(s) \\ 0 & e^{-sh_1}I_{1\sim N} \end{bmatrix}\bar{T}_{2,\mathrm{o}}^{-1}(s)\right)^{\sim} T_1(s)\right\}_+$$

$$= \left\{\bar{T}_{2,\mathrm{o}}^{-\sim}(s)\begin{bmatrix} I_0 & 0 \\ R^{\sim}(s) - e^{sh_1}\Pi_{12}^{\sim}(s)\Pi_{11}^{-\sim}(s) & e^{sh_1}I_{1\sim N} \end{bmatrix} T_2^{\sim}(s)T_1(s)\right\}_+.$$

where

$$I_{1\sim N} = \mathrm{diag}(I_1, \ldots, I_N).$$

The last expression shows that the elements of $T_{2,\mathrm{i}}^{\sim}(s)T_1(s)$ are either a rational transfer function or a rational transfer function multiplied by e^{sh_1}. Both have a rational stable part. $\qquad\square$

The proof of Theorem 4.8 results in the solution of a special case of the one-sided regulator problem, where the delay operator only have at most two distinct delays.

Corollary 4.9 (Solution to the simple delay operator case). *Consider a special case of the regulator problem* (4.30) *problem in which the delay operator $\Lambda_\bullet(s) = \Lambda_s(s)$ is a simple delay operator of the form* (4.20) *containing at most two distinct delays. Let $T_{2,o}(s)$ be defined such that $\Lambda_s^\sim(s)T_2^\sim(s)T_2(s)\Lambda_s(s) = T_{2,o}^\sim(s)T_{2,o}(s)$ with $T_{2,o}(s)$ bistable. Then the H_2 optimal controller that minimizes*

$$\|T_1(s) + T_2(s)\Lambda_s(s)K(s)\|_2$$

is given by

$$K_{opt}(s) = -T_{2,o}^{-1}(s)\left\{T_{2,i}^\sim(s)T_1(s)\right\}_+ \tag{4.51}$$

where $T_{2,i}(s) = T_2(s)\Lambda_s(s)T_{2,o}^{-1}(s)$.

Remark 4.10 (One sided regulator problem solution). Theorem 4.8 and Corollary 4.9 provides the complete solution of the one-sided regulator problem. By applying Theorem 4.8 $(N-1)$ times, where N is the number of nonzero delays in the delay operator, the problem is reduced to the one having a simple delay operator of the form (4.20), which in turn may be solved using Corollary 4.9.

4.5. Solution to the two-sided regulator problem

Now let us return to solving the two-sided regulator problem (Problem 4.3). The problem is minimizing $\|T_1(s)+T_2(s)\Lambda_u(s)K(s)\Lambda_y(s)T_3(s)\|_2$ over $K(s)$ in H_2 where conditions A4, A5, and A6 of Section 4.3 are satisfied. Suppose that the delay operators $\Lambda_u(s)$ and $\Lambda_y(s)$ have N and M nonzero distinct delays, respectively. The solution consists of two stages, the first of which deals with $T_2(s)\Lambda_u(s)$ and the second deals with $\Lambda_y(s)T_3(s)$. The following provides a sketch of the solution.

We begin by absorbing $\Lambda_y(s)T_3 v$ into the controller, transforming the problem to

$$\min_{K_u \in H_2} \|T_1(s) + T_2(s)\Lambda_u(s)K_u(s)\|_2$$

under the condition

$$K_u(s) = K(s)\Lambda_y(s)T_3(s), \ K(s) \in H_2.$$

This problem is a one-sided problem, to which we may apply Theorem 4.8. By applying this theorem N times, the problem is reduced to

$$\min_{K_u \in H_2} \|\tilde{T}_1(s) + \tilde{T}_2(s)\Psi(s)K_u(s)\|_2$$

for certain rational transfer functions $\tilde{T}_1(s) \in H_2$, $\tilde{T}_2(s) \in H_\infty$, and a certain non-rational transfer function $\Psi(s) \in H_\infty$. Moreover, $\tilde{T}_2(s)$ and $\Psi(s)$ are in fact bistable. By substituting the condition $K_u(s) = K(s)\Lambda_y(s)T_3(s)$, $K(s) \in H_2$ back into the regulator problem and absorb the bistable factor $(\tilde{T}_2(s)\Psi(s))$ into the controller, we obtain the problem

$$\min_{K_y \in H_2} \|\tilde{T}_1(s) + K_y(s)\Lambda_y(s)T_3(s)\|_2 = \min_{K_y^T \in H_2} \|\tilde{T}_1^T(s) + T_3^T(s)\Lambda_y(s)K_y^T(s)\|_2$$

where $K_y(s) = \tilde{T}_2(s)\Psi(s)K(s)$. The latter problem is a one-sided regulator problem and may be solved by applying Theorem 4.8 $(M-1)$ times followed by application of Corollary 4.9.

The above discussion is formally stated in the following theorem.

Theorem 4.11 (Two sided regulator problem solution). *Consider the problem of minimizing* $\|T_1(s) + T_2(s)\Lambda_u(s)K(s)\Lambda_y(s)T_3(s)\|_2$ *over* $K(s) \in H_2$ *where* $T_1(s)$, $T_2(s)$, *and* $T_3(s)$ *are rational transfer functions satisfying the assumptions A4, A5, and A6 of Section 4.3. Suppose that* $\Lambda_u(s)$ *and* $\Lambda_y(s)$ *have N and M nonzero distinct delays, respectively. Then the optimal controller may be obtained using the following algorithm:*

1. *Absorb* $\Lambda_y(s)T_3(s)$ *into the controller, transforming the problem to*

$$\min_{K_u \in H_2} \|T_1(s) + T_2(s)\Lambda_u(s)K_u(s)\|_2 \tag{4.52}$$

   under the condition

$$K_u(s) = K(s)\Lambda_y(s)T_3(s), \ K(s) \in H_2. \tag{4.53}$$

2. *Apply Theorem 4.8 N times, so that the problem* (4.52) *reduces to the regulator problem*

$$\min_{K_u \in H_2} \|\tilde{T}_1(s) + \tilde{T}_2(s)\Psi(s)K_u(s)\|_2 \tag{4.54}$$

   for certain rational transfer functions $\tilde{T}_1(s) \in H_2$, $\tilde{T}_2(s) \in H_\infty$, *and a certain non-rational transfer function* $\Psi(s) \in H_\infty$ *with* $\tilde{T}_2^{-1}(s), \Psi^{-1}(s) \in H_\infty$.

3. *Substitute the condition* (4.53) *back into the regulator problem* (4.54) *and absorb the bistable factor* $(\tilde{T}_2(s)\Psi(s))$ *into the controller, which results in the problem*

$$\min_{K_y \in H_2} \|\tilde{T}_1(s) + K_y(s)\Lambda_y(s)T_3(s)\|_2 = \min_{K_y^T \in H_2} \|\tilde{T}_1^T(s) + T_3^T(s)\Lambda_y(s)K_y^T(s)\|_2 \tag{4.55}$$

   where there is a stable bijection between K *and* K_y *governed by the equation*

$$K_y(s) = \tilde{T}_2(s)\Psi(s)K(s). \tag{4.56}$$

4. *Apply Theorem 4.8 $(M-1)$ times to the right hand side of* (4.55) *to obtain a regulator problem with simple delay operator containing at most two distinct delays. Solve the latter problem using Corollary 4.9.*

4.6. Numerical example

To illustrate how the algorithm Theorem 4.11 is utilized to solve a two-sided regulator problem, a numerical example is presented in this section. The example is an extended version of a filtering problem in [Kui03].

Consider the two sided regulator setup of Figure 4.2, where the rational blocks are given by

$$T_1(s) = \begin{bmatrix} \frac{\beta}{s+1} & 0 \\ 0 & \frac{\gamma}{s+1} \end{bmatrix}, \quad T_2(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T_3(s) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad (4.57)$$

and the delay operators are

$$\Lambda_u(s) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-sh_u} \end{bmatrix}, \quad \Lambda_y(s) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-sh_y} \end{bmatrix}. \qquad (4.58)$$

The problem is

$$\min_{K \in H_2} \|T_1(s) + T_2(s)\Lambda_u(s)K(s)\Lambda_y(s)T_3(s)\|_2. \qquad (4.59)$$

Theorem 4.11 provides the algorithm to solve this problem:

- absorb $\Lambda_y(s)T_3(s)$ into $K(s)$ so that the problem becomes

$$\min_{K_u \in H_2} \|T_1(s) + T_2(s)\Lambda_u(s)K_u(s)\|_2 \qquad (4.60)$$

  under the condition

$$K_u(s) = K(s)\Lambda_y(s)T_3(s), \qquad (4.61)$$

- apply the delay reduction theorem (Theorem 4.8) to the single-sided problem (4.60), which results in the problem

$$\min_{K_u \in H_2} \|\tilde{T}_1(s) + \tilde{T}_2(s)\Psi(s)K_u(s)\|_2 \qquad (4.62)$$

  for certain rational transfer functions $\tilde{T}_1(s)$ and $\tilde{T}_2(s)$, and a non-rational $\Psi(s)$, with $T_2(s)$ and $\Psi(s)$ bistable,

- substitute the condition (4.61) back into the problem (4.62) to obtain

$$\min_{K_y \in H_2} \|\tilde{T}_1(s) + K_y(s)\Lambda_y(s)T_3(s)\|_2 = \min_{K_y^T \in H_2} \|\tilde{T}_1^T(s) + T_3^T(s)\Lambda_y(s)K_y^T(s)\|_2, \tag{4.63}$$

  where $K_y(s) = \tilde{T}_2(s)\Psi(s)K(s)$,

- solve the problem (4.63) and recover $K_{\text{opt}}(s)$:

$$K_{\text{opt}}(s) = \Psi^{-1}(s)\tilde{T}_2^{-1}(s)K_{y,\text{opt}}(s). \tag{4.64}$$

We begin executing the algorithm by applying the delay reduction theorem to the problem $\min_{K_u \in H_2} \|\tilde{T}_1(s) + \tilde{T}_2(s)\Psi(s)\Lambda_u(s)K_u(s)\|_2$. First we need to compute the bistable spectral factor $T_{2,\text{o}}(s)$ such that

$$\Lambda_u^\sim(s)T_2^\sim(s)T_2(s)\Lambda_u(s) = T_{2,\text{o}}^\sim(s)T_{2,\text{o}}(s).$$

We proceed by using the formulas of Lemma 4.6. First we compute

$$\Pi(s) = T_2^\sim(s)T_2(s) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \tag{4.65}$$

We choose

$$\Phi_u(s) = -e^{-sh_u}\Pi_{11}^{-1}(s)\Pi_{12}(s) = -e^{-sh_u}, \tag{4.66}$$
$$R(s) = 0. \tag{4.67}$$

Observe that $\Phi_u(s)$ and $R(s)$ are stable and rational, respectively, and satisfy (4.22). Using (4.23) we compute

$$\begin{aligned} W(s) &= \begin{bmatrix} 1 & 0 \\ \Phi_u^\sim(s) & 1 \end{bmatrix} \Lambda_u^\sim(s)T_2^\sim(s)T_2(s)\Lambda_u(s) \begin{bmatrix} 1 & \Phi_u(s) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{4.68}$$

Since the bistable

$$\bar{T}_{2,\text{o}}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.69}$$

trivially satisfy $W(s) = \bar{T}_{2,\text{o}}^\sim(s)\bar{T}_{2,\text{o}}(s)$, it follows that a bistable spectral factor $T_{2,\text{o}}(s)$ is given by

$$T_{2,\text{o}}(s) = \bar{T}_{2,\text{o}}(s) \begin{bmatrix} 1 & -\Phi_u(s) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & e^{-sh_u} \\ 0 & 1 \end{bmatrix}. \tag{4.70}$$

This results in the inner-outer factorization $T_2(s)\Lambda_u(s) = T_{2,\text{o}}(s)T_{2,\text{i}}(s)$ with

$$T_{2,\text{i}}(s) = T_2(s)\Lambda_u(s)T_{2,\text{o}}^{-1}(s) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-sh_u} \end{bmatrix}. \tag{4.71}$$

Now, we proceed by computing $\tilde{T}_1(s)$, $\tilde{T}_2(s)$, and $\Psi(s)$ using (4.48,4.49,4.50):

$$\tilde{T}_1(s) = \{T_{2,\mathrm{i}}^\sim(s)T_1(s)\}_+ = \left\{ \begin{bmatrix} \frac{\beta}{s+1} & 0 \\ 0 & \frac{\gamma e^{sh_u}}{s+1} \end{bmatrix} \right\}_+ = \begin{bmatrix} \frac{\beta}{s+1} & 0 \\ 0 & \frac{\gamma e^{-h_u}}{s+1} \end{bmatrix}, \quad (4.72)$$

$$\tilde{T}_2(s) = \bar{T}_{2,\mathrm{o}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (4.73)$$

$$\Psi(s) = \begin{bmatrix} 1 & -\Phi_u(s) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & e^{-sh_u} \\ 0 & 1 \end{bmatrix}. \quad (4.74)$$

Substituting the condition (4.61) back into the problem (4.62) and take transpose, we obtain the problem $\min_{K_y^T \in H_2} \|\tilde{T}_1^T(s) + T_3^T(s)\Lambda_y(s)K_y^T(s)\|_2$. To solve this problem, first we need to compute the bistable spectral factor $T_{3,\mathrm{o}}$ satisfying

$$\Lambda_y^\sim(s)(T_3^T)^\sim(s)T_3^T(s)\Lambda_y(s) = T_{3,\mathrm{o}}^\sim(s)T_{3,\mathrm{o}}(s).$$

Since $T_3^T(s) = T_2(s)$ and the delay operator $\Lambda_y(s)$ is structurally identical to $\Lambda_u(s)$, we readily obtain the spectral factor:

$$T_{3,\mathrm{o}}(s) = \begin{bmatrix} 1 & e^{-sh_y} \\ 0 & 1 \end{bmatrix}. \quad (4.75)$$

It follows that the inner factor $T_{3,\mathrm{i}}(s)$ satisfying $T_3^T(s)\Lambda_y(s) = T_{3,\mathrm{o}}(s)T_{3,\mathrm{i}}(s)$ is given by

$$T_{3,\mathrm{i}}(s) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-sh_y} \end{bmatrix}. \quad (4.76)$$

Furthermore we have that

$$\{T_{3,\mathrm{i}}^\sim(s)\tilde{T}_1(s)\}_+ = \left\{ \begin{bmatrix} \frac{\beta}{s+1} & 0 \\ 0 & \frac{\gamma e^{sh_y}e^{-h_u}}{s+1} \end{bmatrix} \right\}_+ = \begin{bmatrix} \frac{\beta}{s+1} & 0 \\ 0 & \frac{\gamma e^{-(h_u+h_y)}}{s+1} \end{bmatrix}, \quad (4.77)$$

and the optimal controller $K_{y,\mathrm{opt}}^T(s)$ of the single-sided problem (4.63) may obtained using (4.51) of Corollary 4.9:

$$K_{y,\mathrm{opt}}^T(s) = -T_{3,\mathrm{o}}^{-1}(s)\{T_{3,\mathrm{i}}^\sim(s)\tilde{T}_1(s)\}_+,$$

resulting in

$$K_{y,\mathrm{opt}}(s) = \begin{bmatrix} \frac{-\beta}{s+1} & 0 \\ \frac{\gamma e^{-sh_y}e^{-(h_u+h_y)}}{s+1} & \frac{-\gamma e^{-(h_u+h_y)}}{s+1} \end{bmatrix}. \quad (4.78)$$

Finally, the optimal controller of the original two sided regulator (4.59) problem may be recovered:

$$K_{\mathrm{opt}}(s) = \Psi^{-1}(s)\tilde{T}_2^{-1}(s)K_{y,\mathrm{opt}}(s)$$

$$= \begin{bmatrix} -\frac{\beta}{s+1} - \frac{\gamma e^{-s(h_u+h_y)}e^{-(h_u+h_y)}}{s+1} & \frac{\gamma e^{-sh_u}e^{-(h_u+h_y)}}{s+1} \\ \frac{\gamma e^{-sh_y}e^{-(h_u+h_y)}}{s+1} & -\frac{\gamma e^{-(h_u+h_y)}}{s+1} \end{bmatrix}. \quad (4.79)$$

Due to the structure of the problem, the (1,1) block of $T_1(s)$, i.e. $\beta/(s+1)$, can be canceled. Hence, the optimal H_2-norm is independent of β. To get a feeling of how much the H_2-norm varies as the delays vary, we examine two extreme cases. For zero delays ($h_u = h_y = 0$), it may be verified that

$$K_{\text{opt}}(s) = \begin{bmatrix} -\frac{\beta}{s+1} - \frac{\gamma}{s+1} & \frac{\gamma}{s+1} \\ \frac{\gamma}{s+1} & -\frac{\gamma}{s+1} \end{bmatrix}$$

resulting in

$$T_1(s) + T_2(s)\Lambda_u(s)K_{\text{opt}}(s)\Lambda_y(s)T_3(s) = 0,$$

and thus the optimal H_2-norm is also zero. For infinite delays ($h_u = h_y = \infty$), it may be verified that

$$K_{\text{opt}}(s) = \begin{bmatrix} -\frac{\beta}{s+1} & 0 \\ 0 & 0 \end{bmatrix}$$

so that

$$T_1(s) + T_2(s)\Lambda_u(s)K_{\text{opt}}(s)\Lambda_y(s)T_3(s) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\gamma}{s+1} \end{bmatrix}.$$

The associated squared optimal H_2-norm is $\frac{1}{2}\gamma^2$. Hence as the delays vary, the squared optimal H_2 norm varies from zero to $\frac{1}{2}\gamma^2$.

4.7. State space formulas of the spectral factorization

The solution of the two sided regulator problem discussed in Section 4.5 shows that solving the problem boils down to the computation of the spectral factorization of delayed transfer functions with at most two distinct delays. Lemma 4.6, which is based on results in [MZ00], provides the formulas for the computation. However, the formulas, especially (4.23), are rather involved. In this section, an alternative state space proof utilizing what is called the Schur transformation is provided. The proof leads to a simpler state space formulation of the spectral factors, which makes computation of the optimal controller easier. This section is based on results in [Kui03], which in turn largely stem from [MMZ02].

4.7.1. Schur transformation

The proof and the resulting formulas are significantly simplified through the use of what is called upper Schur transformation (see [Kim96] and [MMZ02]). It is defined in what follows.

Definition 4.12 (Upper Schur transformation). For a 2×2 block transfer function

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix},$$

the *upper Schur transformation* is defined as

$$S_u(H(s)) = \begin{bmatrix} H_{11}^{-1}(s) & -H_{11}^{-1}(s)H_{12}(s) \\ H_{21}(s)H_{11}^{-1}(s) & H_{22}(s) - H_{21}(s)H_{11}^{-1}(s)H_{12} \end{bmatrix}, \tag{4.80}$$

assuming that $H_{11}^{-1}(s)$ exists.

The upper Schur transformation may interpreted as swapping the upper part of the inputs and output:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = H \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} u_1 \\ y_2 \end{bmatrix} = S_u(H) \begin{bmatrix} y_1 \\ u_2 \end{bmatrix}. \tag{4.81}$$

The above interpretation leads to a simple derivation of a state space representation of the transformed transfer function. Suppose that $H(s)$ has a state space realization of the form

$$H(s) = \left[\begin{array}{c|cc} A_H & B_{H1} & B_{H2} \\ \hline C_{H1} & D_{H11} & D_{H12} \\ C_{H2} & D_{H21} & D_{H22} \end{array} \right], \tag{4.82}$$

then $S_u(H(s))$ has the following realization:

$$
\begin{aligned}
&S_u(H(s)) \\
&= \left[\begin{array}{c|cc} A_H - B_{H1}D_{H11}^{-1}C_{H1} & B_{H1}D_{H11}^{-1} & B_{H2} - B_{H1}D_{H11}^{-1}D_{H12} \\ \hline -D_{H11}^{-1}C_{H1} & D_{H11}^{-1} & -D_{H11}^{-1}D_{H12} \\ C_{H2} - D_{H21}D_{H11}^{-1}C_{H1} & D_{H21}D_{H11}^{-1} & D_{H22} - D_{H21}D_{H11}^{-1}D_{H12} \end{array} \right].
\end{aligned}
\tag{4.83}
$$

A useful property of the upper Schur transformation is that for arbitrary transfer functions $\Phi_1(s)$, $\Phi_2(s)$, and $H(s)$, it may be verified that

$$S_u\left(\begin{bmatrix} I & 0 \\ \Phi_1(s) & I \end{bmatrix} H(s) \begin{bmatrix} I & \Phi_2(s) \\ 0 & I \end{bmatrix} \right) = S_u(H(s)) + \begin{bmatrix} 0 & -\Phi_2(s) \\ \Phi_1(s) & 0 \end{bmatrix}. \tag{4.84}$$

4.7.2. State space proof of Lemma 4.6

Suppose we are given a stable rational transfer matrix $F(s)$ having full column rank on $j\mathbb{R} \cup \infty$ with a realization of the form

$$F(s) = \left[\begin{array}{c|c} A_F & B_F \\ \hline C_F & D_F \end{array} \right], \tag{4.85}$$

and a delay operator containing at most two distinct delays of the form

$$\Lambda_{\mathrm{s}}(s) = \begin{bmatrix} e^{-sh_1}I_1 & 0 \\ 0 & e^{-sh_2}I_2 \end{bmatrix}, \quad h_2 \geq h_1 \geq 0. \tag{4.86}$$

The problem is to find a bistable transfer function $F_{\mathrm{o}}(s)$ such that

$$\Lambda_{\mathrm{s}}^{\sim}(s)F^{\sim}(s)F(s)\Lambda_{\mathrm{s}}(s) = F_{\mathrm{o}}^{\sim}(s)F_{\mathrm{o}}(s). \tag{4.87}$$

We begin by defining the rational transfer function

$$\Pi(s) = F^{\sim}(s)F(s). \tag{4.88}$$

It may be shown that $\Pi(s)$ has the following state space realization:

$$\Pi(s) = \left[\begin{array}{cc|c} A_F & 0 & B_F \\ -C_F^T C_F & -A_F^T & -C_F^T D_F \\ \hline D_F^T C_F & B_F^T & D_F^T D_F \end{array} \right]. \tag{4.89}$$

Furthermore, by partitioning $\Pi(s)$ compatible with the delay operator $\Lambda_{\mathrm{s}}(s)$, we may write

$$\Pi(s) = \begin{bmatrix} \Pi_{11}(s) & \Pi_{12}(s) \\ \Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_\Pi & B_{\Pi 1} & B_{\Pi 2} \\ \hline C_{\Pi 1} & D_{\Pi 11} & D_{\Pi 12} \\ C_{\Pi 2} & D_{\Pi 21} & D_{\Pi 22} \end{array} \right], \tag{4.90}$$

so that

$$\Lambda_{\mathrm{s}}^{\sim}(s)F^{\sim}(s)F(s)\Lambda_{\mathrm{s}}(s) = \begin{bmatrix} \Pi_{11}(s) & e^{-sh}\Pi_{12}(s) \\ e^{sh}\Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix}, \quad h = h_2 - h_1. \tag{4.91}$$

Clearly, (4.91) is non rational. It will be shown that the non-rational parts of (4.91) may be removed by multiplying it with a bistable factor of the form $\begin{bmatrix} I_1 & \Phi_F(s) \\ 0 & I_2 \end{bmatrix}$. The task now is to find the suitable bistable factor such that

$$W(s) = \begin{bmatrix} I_1 & \Phi_F(s) \\ 0 & I_2 \end{bmatrix}^{\sim} \begin{bmatrix} \Pi_{11}(s) & e^{-sh}\Pi_{12}(s) \\ e^{sh}\Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix} \begin{bmatrix} I_1 & \Phi_F(s) \\ 0 & I_2 \end{bmatrix} \tag{4.92}$$

is rational. This is where the upper Schur transformation and the property (4.84) play their role. First define

$$\Omega(s) = S_{\mathrm{u}}(\Pi(s)) = \begin{bmatrix} \Pi_{11}^{-1}(s) & -\Pi_{11}^{-1}(s)\Pi_{12}(s) \\ \Pi_{21}(s)\Pi_{11}^{-1}(s) & \Pi_{22}(s) - \Pi_{21}(s)\Pi_{11}^{-1}(s)\Pi_{12} \end{bmatrix}. \tag{4.93}$$

The transfer function $\Omega(s)$ has the following state space realization

$$
\Omega(s) = \begin{bmatrix} \Omega_{11}(s) & \Omega_{12}(s) \\ \Omega_{21}(s) & \Omega_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_\Omega & B_{\Omega 1} & B_{\Omega 2} \\ \hline C_{\Omega 1} & D_{\Omega 11} & D_{\Omega 12} \\ C_{\Omega 2} & D_{\Omega 21} & D_{\Omega 22} \end{array} \right]
$$
$$
= \left[\begin{array}{c|cc} A_\Pi - B_{\Pi 1} D_{\Pi 11}^{-1} C_{\Pi 1} & B_{\Pi 1} D_{\Pi 11}^{-1} & B_{\Pi 2} - B_{\Pi 1} D_{\Pi 11}^{-1} D_{\Pi 12} \\ \hline -D_{\Pi 11}^{-1} C_{\Pi 1} & D_{\Pi 11}^{-1} & -D_{\Pi 11}^{-1} D_{\Pi 12} \\ C_{\Pi 2} - D_{\Pi 21} D_{\Pi 11}^{-1} C_{\Pi 1} & D_{\Pi 21} D_{\Pi 11}^{-1} & D_{\Pi 22} - D_{\Pi 21} D_{\Pi 11}^{-1} D_{\Pi 12} \end{array} \right]. \tag{4.94}
$$

Applying the upper Schur transformation to (4.91), we obtain

$$
\begin{aligned}
S_{\mathrm{u}}(W(s)) &= S_{\mathrm{u}} \left(\begin{bmatrix} I_1 & 0 \\ \Phi_{\widetilde{F}}(s) & I_2 \end{bmatrix} \begin{bmatrix} \Pi_{11}(s) & e^{-sh}\Pi_{12}(s) \\ e^{sh}\Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix} \begin{bmatrix} I_1 & \Phi_F(s) \\ 0 & I_2 \end{bmatrix} \right) \\
&= S_{\mathrm{u}} \left(\begin{bmatrix} \Pi_{11}(s) & e^{-sh}\Pi_{12}(s) \\ e^{sh}\Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix} \right) + \begin{bmatrix} 0 & -\Phi_F(s) \\ \Phi_{\widetilde{F}}(s) & 0 \end{bmatrix} \\
&= \begin{bmatrix} \Pi_{11}^{-1}(s) & -e^{-sh}\Pi_{11}^{-1}(s)\Pi_{12}(s) \\ e^{sh}\Pi_{21}(s)\Pi_{11}^{-1}(s) & \Pi_{22}(s) - \Pi_{21}(s)\Pi_{11}^{-1}(s)\Pi_{12} \end{bmatrix} + \begin{bmatrix} 0 & -\Phi_F(s) \\ \Phi_{\widetilde{F}}(s) & 0 \end{bmatrix} \\
&= \begin{bmatrix} \Omega_{11}(s) & e^{-sh}\Omega_{12}(s) \\ e^{sh}\Omega_{21}(s) & \Omega_{22}(s) \end{bmatrix} + \begin{bmatrix} 0 & -\Phi_F(s) \\ \Phi_{\widetilde{F}}(s) & 0 \end{bmatrix} \\
&= \begin{bmatrix} \Omega_{11}(s) & e^{-sh}\Omega_{12}(s) - \Phi_F(s) \\ e^{sh}\Omega_{21}(s) + \Phi_{\widetilde{F}}(s) & \Omega_{22}(s). \end{bmatrix}
\end{aligned} \tag{4.95}
$$

To make $\Omega(s)$, and hence $W(s)$, rational, we may take

$$
-\Phi_F(s) = \pi_h(e^{-sh}\Omega_{12}(s)) = \left[\begin{array}{c|c} A_\Omega & B_{\Omega 2} \\ \hline C_{\Omega 1} e^{-A_\Omega h} & 0 \end{array} \right] - e^{-sh} \left[\begin{array}{c|c} A_\Omega & B_{\Omega 2} \\ \hline C_{\Omega 1} & D_{\Omega 12} \end{array} \right]. \tag{4.96}
$$

Since $\Omega_{12}(s) = -\Pi_{11}^{-1}(s)\Pi_{12}(s)$, $\Phi_F(s)$ in (4.96) satisfies Equation (4.22) of Lemma 4.6.

Equation (4.96) also implies that we may take

$$
R(s) = e^{-sh}\Omega_{12}(s) - \Phi_F(s) = \left[\begin{array}{c|c} A_\Omega & e^{-A_\Omega h} B_{\Omega 2} \\ \hline C_{\Omega 1} & 0 \end{array} \right], \tag{4.97}
$$

and it may be shown that

$$
-R^{\sim}(s) = e^{sh}\Omega_{21}(s) + \Phi_{\widetilde{F}}(s) = \left[\begin{array}{c|c} A_\Omega & B_{\Omega 1} \\ \hline C_{\Omega 2} e^{A_\Omega h} & 0 \end{array} \right], \tag{4.98}
$$

so that we may obtain the state space realization of $S_{\mathrm{u}}(W(s))$:

$$
\begin{aligned}
S_{\mathrm{u}}(W(s)) &= \begin{bmatrix} \Omega_{11}(s) & R(s) \\ -R^\sim(s) & \Omega_{22}(s) \end{bmatrix} \\
&= \left[\begin{array}{c|cc} A_\Omega & B_{\Omega1} & e^{-A_\Omega h}B_{\Omega2} \\ \hline C_{\Omega1} & D_{\Omega11} & 0 \\ C_{\Omega2}e^{A_\Omega h} & 0 & D_{\Omega22} \end{array} \right].
\end{aligned}
\tag{4.99}
$$

It follows that the state space realization of $W(s)$ is given by

$$
\begin{aligned}
W(s) &= S_{\mathrm{u}}(S_{\mathrm{u}}(W(s))) \\
&= \left[\begin{array}{c|cc} A_\Omega - B_{\Omega1}D_{\Omega11}^{-1}C_{\Omega1} & B_{\Omega1}D_{\Omega11}^{-1} & e^{-A_\Omega h}B_{\Omega2} \\ \hline -D_{\Omega11}^{-1}C_{\Omega1} & D_{\Omega11}^{-1} & 0 \\ C_{\Omega2}e^{A_\Omega h} & 0 & D_{\Omega22} \end{array} \right] \\
&= \left[\begin{array}{c|cc} A_\Pi & B_{\Pi1} & e^{-A_\Omega h}(B_{\Pi2} - B_{\Pi1}D_{\Pi11}^{-1}D_{\Pi12}) \\ \hline C_{\Pi1} & D_{\Pi11} & 0 \\ (C_{\Pi2} - D_{\Pi21}D_{\Pi11}^{-1}C_{\Pi1})e^{A_\Omega h} & 0 & D_{\Pi22} - D_{\Pi21}D_{\Pi11}^{-1}D_{\Pi12} \end{array} \right],
\end{aligned}
\tag{4.100}
$$

where $A_\Omega = A_\Pi - B_{\Pi1}D_{\Pi11}^{-1}C_{\Pi1}$. It is clear that $W(s)$ is rational and proper, and since it shares the same A-matrix with $\Pi(s)$, it also has no poles in the imaginary axis.

At this point, what is left is to show that it is possible to construct a bistable factor $\bar{F}_{\mathrm{o}}$ that satisfies

$$
W(s) = \bar{F}_{\mathrm{o}}^\sim(s)\bar{F}_{\mathrm{o}},
\tag{4.101}
$$

so that the spectral factorization of $F(s)\Lambda_{\mathrm{s}}(s)$ is given by

$$
\Lambda_{\mathrm{s}}^\sim(s)F^\sim(s)F(s)\Lambda_{\mathrm{s}}(s) = F_{\mathrm{o}}^\sim(s)F_{\mathrm{o}}(s),
\tag{4.102}
$$

where

$$
F_{\mathrm{o}} = \bar{F}_{\mathrm{o}}(s)\begin{bmatrix} I_1 & -\Phi_F(s) \\ 0 & I_2 \end{bmatrix}.
\tag{4.103}
$$

To this end we need a result from [AV73] stipulating that there exist a real rational, bistable $\bar{F}_{\mathrm{o}}(s)$ such that $W(s) = \bar{F}_{\mathrm{o}}^\sim(s)\bar{F}_{\mathrm{o}}$ if and only if $W(s)$ satisfies the following condition:

- $W(s)$ is real rational,

- $W(s) = W^\sim(s)$,

- $W(s)$ has no poles in the imaginary axis, and

- $W(s) > 0$ on $j\mathbb{R} \cup \infty$.

The first condition is clearly satisfied, while by construction the second and third condition are also satisfied. By assumption $F(s)$ has full column rank on $j\mathbb{R} \cup \infty$ and $\Lambda_s(s)$ is nonsingular on $j\mathbb{R}$. Hence, $\Lambda_s^{\sim}(s)\Pi(s)\Lambda_s(s)$ is positive definite on $j\mathbb{R}$. Combined with the fact that $\begin{bmatrix} I_1 & -\Phi_F(s) \\ 0 & I_2 \end{bmatrix}$ is nonsingular, it follows that $W(s)$ is positive definite on $j\mathbb{R}$. To show that $W(s)$ is also positive definite at $s = \infty$, recall that $R(s)$ is strictly proper, implying that $R(s)|_{s=\infty} = 0$ and

$$
\begin{aligned}
W(s)|_{s=\infty} &= \left(\begin{bmatrix} \Pi_{11}(s) & 0 \\ 0 & \Pi_{22}(s) - \Pi_{21}(s)\Pi_{11}^{-1}(s)\Pi_{12}(s) \end{bmatrix} \right)\Bigg|_{s=\infty} \\
&= \left(\begin{bmatrix} I & 0 \\ -\Pi_{21}(s)\Pi_{11}^{-1}(s) & I \end{bmatrix} \Pi(s) \begin{bmatrix} I & -\Pi_{11}^{-1}(s)\Pi_{21}^{\sim}(s) \\ 0 & I \end{bmatrix} \right)\Bigg|_{s=\infty}.
\end{aligned}
\tag{4.104}
$$

The fact that

$$
\Pi(s)|_{s=\infty} = (F^{\sim}(s)F(s))|_{s=\infty} > 0
$$

and

$$
\begin{bmatrix} I & 0 \\ -\Pi_{21}(s)\Pi_{11}^{-1}(s) & I \end{bmatrix}\Bigg|_{s=\infty}
$$

is non singular implies that $W(s)|_{s=\infty}$ is positive definite.

We have shown that it is possible to construct the spectral factorization of $W(s)$. For explicit state space formulas of the spectral factor $\bar{F}_o(s)$ satisfying $W(s) = \bar{F}_o^{\sim}(s)\bar{F}_o$, see [MM05a].

4.8. Concluding remarks

This chapter proposes a solution of the standard H_2-optimal control problem of systems with multiple i/o delays using frequency domain methods. The approach is to convert the standard problem to the two sided regulator problem, which is solved using spectral factorization and special decomposition of the delay operator. Compared to the time domain approach of Chapter 3, this approach has an obvious advantage of being able to handle the case with delays occurring on both sides of the controller. However, the frequency domain method also has its drawbacks.

The method only works for systems that are open-loop stable. In its current form, it cannot handle unstable plants. Furthermore, the method seems to result in a controller in which the rational part has unnecessarily high state dimension. These problems are not observed in the time domain solution, where unstable plants can be handled as easily as the stable ones and the rational part of the controller retains the state dimension of the plant.

Therefore, unless delays occur on both sides of the controller, it is advisable to avoid using the frequency domain solution and use the time domain solution instead.

4. The Standard H_2 Problem: Frequency Domain Approach

5

H_2 Control of Preview Systems

After dealing with systems with delays in the preceding chapters, in this chapter we touch a different but closely related topic: preview systems. In certain control problems, all or parts of the external input signals are known in advance. An example is a tracking problem where the tracked trajectory is known in advance. Exploiting this knowledge might improve the control system performance. However, most controller design techniques do not take it into account. Control systems that do exploit the advance knowledge of the input are commonly designated as preview control systems. In this chapter, we show that the ideas from Chapter 3 may be applied to solve the H_2-optimal controller for systems with previewed input.

While previewing the input signals may increase the performance of a control system, it also increases the complexity of the controller. For this reason, the performance gain has to be significant enough to justify the increased controller complexity. Therefore, the following questions, which are paraphrased from questions posed in [AM79] in studying the advantage of smoothing over filtering, are highly relevant:

- How does the performance gain owing to the previewed input with preview times vary as the preview times increase?

- What is the maximum achievable performance gain, i.e., what is the performance gain associated with infinite preview times?

Besides the primary objective of obtaining the H_2-optimal controller, it is also the aim of this chapter to answer the above questions. The main results include the formulation of the optimal controller for both the single preview time case and the multiple preview times case. An explicit formula that clearly shows the performance gain owing to the previewed input is also derived. This chapter is based on the paper [MM05b].

5.1. Literature review

Several results have been put forward to incorporate the advance knowledge of the external input signals into H_∞ and H_2 designs. The H_∞ preview
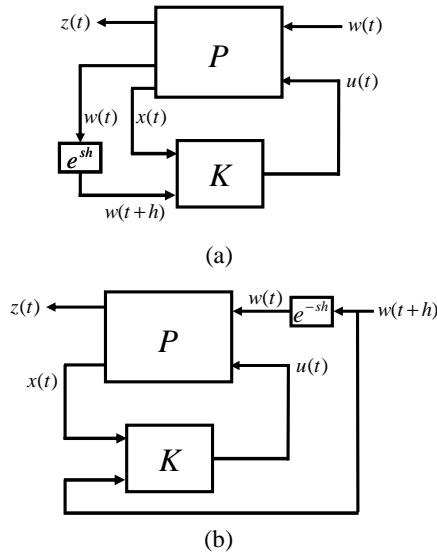
Figure 5.1.: The single preview time control setup (a) and its equivalent (b).

control problem was considered in [KI03b] and [KI03a]. Later, a game theoretic solutions in both continuous and discrete time were proposed in the papers [TM05a] and [TM05b]. The closely related problem of H_∞ fixed-lag smoothing was treated in [Mir03b], [TS94], [TM05a], [TM05b], and [MT04]. In particular, the paper [MT04] provides an explicit computation of the optimal cost as a function of the preview time.

In the field of H_2 design, the H_2 fixed-lag smoothing problem has been solved in the 60s (see [And69], [AM79] and references therein). Lately, there have been several results that treat the H_2 control problem of preview systems.

In the discrete time framework, the paper [MC95] solved the linear quadratic tracking problem with preview in the deterministic setting, while the paper [MZ89] treated the problem in stochastic setting. The latter was extended to cover the minimax version of the problem in [MCG90].

In continuous time framework, the paper [Tom75] solved the linear quadratic optimal tracking problem with preview. The paper [KI99] treated a linear quadratic problem with stored disturbance, while in [Koj04] an H_2 preview control problem, which is equivalent to the single input control problem considered in this chapter, was solved. More recently, the paper [MZ05] treated a feed forward disturbance rejection problem. Both [Koj04] and [MZ05] utilized the same ingredient: partitioning the optimization time interval. In [MZ05], the problem is split into three problems: a finite horizon

LQR problem with assigned final state, an infinite horizon LQR problem, and a problem of selecting the intermediate state. In [Koj04], the problem is split into an infinite horizon problem and a finite horizon problem, where the finite horizon part is solved using orthogonal projection arguments. The idea of splitting the optimization interval was first employed in [Tad97a] in the context of robust control in the gap and is also used for solving the H_2 control problem of systems with multiple i/o delays in Chapter 3 of this thesis.

In this chapter, an alternative derivation of the solution of the H_2 preview feedback control problem is provided. As in [Koj04], the technique of splitting the optimization time interval into two time intervals with the preview time h as the boundary is used. The problem is effectively split into two parts: a standard infinite horizon LQR problem and a finite horizon LQR problem with a non-standard constraint of a jump in the final state. The standard infinite horizon part results in state feedback part of the optimal controller, while the non-standard finite horizon part is tackled using the Pontryagin minimum principle and results in finite impulse response part of the optimal controller.

Both the derivation in this chapter and in [Koj04] result in the same formulation of the optimal controller for the single input case. The main difference lies in the formula for the optimal H_2-norm. The formula in [Koj04] requires solving a differential Riccati equation and does not provide a clear insight in determining the effect of the preview time h on the H_2 performance. The derivation in this chapter results in a different formulation that complements the results in [Koj04]. Not only that the formula derived in this chapter appears simpler (it only requires solving a Lyapunov equation), but it also clearly shows the performance gain owing to the previewed input. It also allows the computation of the maximum achievable performance corresponding to the infinite preview time. Note that for the H_2 fixed-lag smoothing problem, this type of expression that explicitly shows the performance gain owing to the smoothing lag appeared in the paper [And69]. Furthermore, this chapter also treats the multiple preview times case.

5.2. Problem Formulation

The preview control system configuration with a single preview time[1] is shown in Figure 5.1(a). It is very similar to the standard full information control system, in which the controller uses the state x and the external input w as its inputs. The only difference is that the external signal w is available to the controller h time units in advance. This fact is represented in

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[1]The multiple preview time case is formulated and solved later in Section 5.5

Figure 5.1(a) by the time advance operator e^{sh}. To avoid employing a time advance operator, the same effect may be achieved by delaying the external input fed to the plant, while the controller receives the non-delayed version. This setting is shown in Figure 5.1(b). The control problem considered in this chapter is of the equivalent form of Figure 5.1(b) and formally stated in what follows.

Problem 5.1 (Single preview time problem). Consider the control system of Figure 5.1(b) where the dynamics of the plant $P(s)$ are governed by the state space equation:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\
z(t) &= C_1 x(t) + D_2 u(t).
\end{aligned}
\tag{5.1}
$$

and the system parameters satisfy the following standard assumptions:

A1 (C_1, A, B_2) is detectable and stabilizable;

A2 $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_2 \end{bmatrix}$ has full column rank $\forall \omega \in \mathbb{R}$.

In addition to the standard assumptions above, to simplify the formulas it is also assumed that

A3 $D_2^T D_2 = I$ and $C_1^T D_2 = 0$.

Assumption A3 will be relaxed later. The problem is to find a stabilizing control K that minimizes the H_2-norm of the transfer function from w to z.

5.3. Single-input case

In this section the case where B_1 is a column vector, i.e. w is a one dimensional signal, is considered. To signify the difference, a lower case b_1 is used in place of B_1. The H_2-norm of the transfer function from w to z is equal to the L_2-norm of z provided that $w(t)$ is a delta function. Therefore, by setting

$$
w(t) = \delta(t - h),
$$

the H_2 optimization problem may be formulated as an LQR problem with a state jump at $t = h$. At this point, the original objective of designing a full information controller that takes $x(t)$ and $w(t + h)$ as inputs is temporarily set aside. Rather the attention is focused on finding the optimal u such that the L_2-norm of z is minimized given $w(t) = \delta(t - h)$. Later it shall be shown that the optimal control law may be implemented by a full information controller with preview as in Figure 5.1 and thus producing the desired optimal controller.
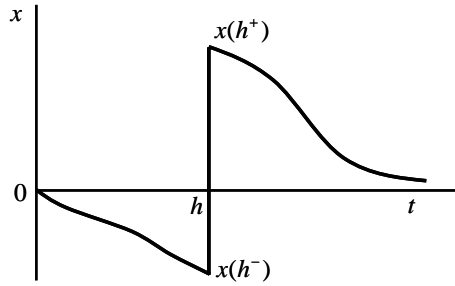
Figure 5.2.: The state trajectory.

Given that $B_1 = b_1$ and $w(t) = \delta(t - h)$, the state space equation (5.1) becomes

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + b_1\delta(t-h) + B_2u(t), \\
z(t) &= C_1x(t) + D_2u(t), \quad x(0) = 0,
\end{aligned} \tag{5.2}$$

and our objective is

$$\begin{aligned}
\min_u J(x_0, u) &= \min_u \int_0^\infty \|C_1x(t) + D_2u(t)\|_2^2 \, dt \\
&= \min_u \int_0^\infty x(t)^T Q x(t) + u(t)^T u(t) dt
\end{aligned} \tag{5.3}$$

where

$$Q = C_1^T C_1. \tag{5.4}$$

The delta function input at $t = h$ in (5.2) raises the state such that $x(h^+) = x(h^-) + b_1$, so that (5.2) may be rewritten as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_2u(t), \\
x(0) &= 0, \ x(h^+) = x(h^-) + b_1, \\
z(t) &= C_1x(t) + D_2u(t).
\end{aligned} \tag{5.5}$$

The state space equation (5.5) together with the criterion function (5.3) constitute an LQR problem. The only difference of the LQR problem (5.5,5.3) and a standard LQR problem is the state jump at $t = h$, as illustrated in in Figure 5.2. One way to circumvent the problem is to use the technique from [MM05c],[Tad97a] of dividing the optimization time horizon into two regions with $t = h$ as the boundary so that the state jump can be considered as a boundary condition. It turns out that the optimal control problem in each time region may be solved essentially independent from the other.

Lemma 5.2 (Finite horizon part). *Consider the LQR problem corresponding to the state space equation (5.5) and the objective (5.3). Suppose that Assumptions A1, A2, and A3 are satisfied. Let M be the stabilizing solution of the familiar LQR Riccati equation*

$$Q + A^T M + M A - M B_2 B_2^T M = 0. \tag{5.6}$$

Define

$$u_{2,opt}(t) = -B_2^T M x(t) \tag{5.7}$$

and let $u_{1,opt}$ be the solution of the LQR problem corresponding to the state-space equation

$$\dot{x}(t) = A x(t) + B_2 u_1(t), \quad x(0) = 0, \tag{5.8}$$

with the objective

$$\min_{u_1} \left((x(h) + b_1)^T M (x(h) + b_1) + \int_0^h x^T Q x + u_1^T u_1 dt \right). \tag{5.9}$$

Then the solution of the LQR problem (5.5,5.3) is given by

$$u_{opt}(t) = [\mathbb{1}(t) - \mathbb{1}(t-h)] u_{1,opt}(t) + \mathbb{1}(t-h) u_{2,opt}(t) \tag{5.10}$$

and the optimal cost is given by (5.9).

Proof. Consider the state space equation (5.5). Assume temporarily that the optimal state at $t = h^-$, denoted by $x_{opt}(h^-)$, is known. It follows that

$$x_{opt}(h^+) = x_{opt}(h^-) + b_1.$$

For the time region $t \in [h^+, \infty]$, the equation (5.5) becomes

$$\dot{x} = A x + B_2 u, \quad x(h^+) = x_{opt}(h^-) + b_1, \tag{5.11}$$

while the cost over this this time region is given by

$$J_{[h^+,\infty]} = \int_h^\infty x(t)^T Q x(t) + u(t)^T u(t) dt. \tag{5.12}$$

The problem of minimizing (5.12) given (5.11) is a standard infinite horizon LQR problem, the solution of which is the state feedback

$$u_{opt}(t) = -B_2^T M x(t), \ t \in [h^+, \infty], \tag{5.13}$$

while the optimal cost is

$$\begin{aligned} J_{[h^+,\infty],opt} &= x(h^+)^T M x(h^+) \\ &= [x_{opt}(h^-) + b_1]^T M [x_{opt}(h^-) + b_1], \end{aligned} \tag{5.14}$$

where M is the solution of the Riccati equation (5.6). Hence, it is proved that for $t \in [h^+, \infty]$ the optimal input is indeed given by the state feedback (5.7). It is also clear that the optimal cost contribution over $t = [h^+, \infty]$, which is given by (5.14), depends solely on $x_{\text{opt}}(h^-)$. It follows that the infinite horizon LQR problem of minimizing (5.3) is equivalent to minimizing the finite horizon cost function

$$\min_u \left(\int_0^h x^T Q x + u^T u dt + [x(h) + b_1]^T M [x(h) + b_1] \right),$$

from which the optimal input for $t \in [0, h^-]$ may be obtained. $\qquad \square$

Lemma 5.2 gives a partial solution to the LQR problem (5.5,5.3). It is now ascertained that for $t \in [h, \infty]$ the optimal input is a state feedback given by (5.7). What is left is to solve the finite horizon LQR problem (5.8,5.9). The solution is summarized in the following lemma.

Lemma 5.3 (Infinite horizon part). *Consider the LQR problem corresponding to the state space equation* (5.8) *with the objective* (5.9) *where M is the stabilizing solution of the Riccati equation* (5.6)*. Then the optimal input of the LQR problem (5.8,5.9) is given by*

$$u_{1,opt}(t) = -B_2^T M x(t) - B_2^T e^{-A_p^T (t-h)} M b_1, \tag{5.15}$$

with $A_p = A - B_2 B_2^T M$.

Proof. We begin by applying the minimum principle[2] to the optimal control problem (5.8,5.9). It may be shown (see for example Appendix C of [AM89]), that the optimal input is given by

$$u_{1,\text{opt}}(t) = B_2^T p(t), \tag{5.16}$$

where the co-state p and the optimal state x satisfy the following equation:

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & B_2 B_2^T \\ Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{5.17}$$

with the boundary condition

$$x(0) = 0, \quad p(h) = -M x(h) - M b_1. \tag{5.18}$$

Notice that except for the boundary condition, the equations are similar to the standard case where $b_1 = 0$. Furthermore, using similar arguments as in the standard case, it may be shown that the differential equation (5.17,5.18) has a unique solution.

---

[2]See Section A.2 of the appendix for a review of the minimum principle.

To obtain the solution of the differential equation (5.17,5.18), define the state transformation

$$q(t) = Mx(t) + p(t). \tag{5.19}$$

With the state transformation and keeping in mind that M is the solution of the Riccati equation (5.6), the differential equation (5.17) is simplified to

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A_p & B_2 B_2^T \\ 0 & -A_p^T \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} \tag{5.20}$$

with the boundary condition

$$x(0) = 0, \quad q(h) = -Mb_1. \tag{5.21}$$

It follows that the trajectory of $q(t)$ is given by

$$q(t) = e^{-A_p^T t} q(0). \tag{5.22}$$

The initial condition $q(0)$ may be computed by setting $t = h$ in (5.22) and substituting the boundary condition (5.21), resulting in:

$$q(0) = -e^{A_p^T h} Mb_1, \tag{5.23}$$

so that the complete expression for $q(t)$ is obtained:

$$q(t) = -e^{-A_p^T (t-h)} Mb_1. \tag{5.24}$$

Using (5.19) and (5.24), $p(t)$ may be computed:

$$p(t) = -Mx(t) - e^{-A_p^T (t-h)} Mb_1. \tag{5.25}$$

The optimal input $u_{1,\text{opt}}(t) = B_2^T p(t)$ is then given by

$$u_{1,\text{opt}}(t) = -B_2^T Mx(t) - B_2^T e^{-A_p^T (t-h)} Mb_1.$$

$$\square$$

The optimal controller

Lemma 5.2 combined with Lemma 5.3 provides a complete solution to the infinite horizon LQR problem (5.5,5.3). By Lemma 5.2 and Lemma 5.3, the optimal u is given by

$$\begin{aligned}
u_{\text{opt}}(t) &= [\mathbb{1}(t) - \mathbb{1}(t-h)]u_{1,\text{opt}}(t) + \mathbb{1}(t-h)u_{2,\text{opt}}(t) \\
&= -[\mathbb{1}(t) - \mathbb{1}(t-h)]B_2^T (Mx(t) + e^{-A_p^T (t-h)} Mb_1) \\
&\quad - \mathbb{1}(t-h)B_2^T Mx(t) \\
&= -B_2^T Mx(t) - [\mathbb{1}(t) - \mathbb{1}(t-h)]B_2^T e^{-A_p^T (t-h)} Mb_1.
\end{aligned} \tag{5.26}$$
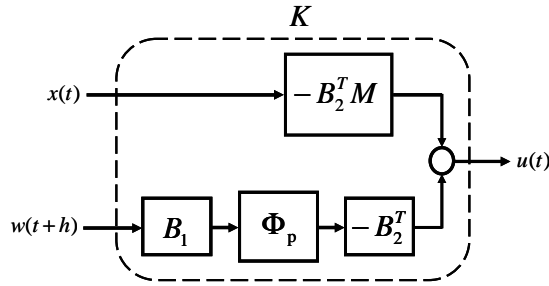
Figure 5.3.: The optimal controller, single preview time case.

This is the unique optimal u for the control system of Figure 5.1 when $w(t) = \delta(t - h)$. Hence, if we manage to find a full information controller with preview that also produces the same input if we set $w(t) = \delta(t - h)$, then we automatically obtain the desired H_2-optimal controller. In the following theorem, the optimal controller is derived.

Theorem 5.4 (Single input optimal controller). *Consider the control system of Figure 5.1(b) where the plant's dynamics are governed by* (5.1)*. Suppose that Assumptions A1, A2, and A3 are satisfied, and that w is a one-dimensional signal, i.e. $B_1 = b_1$ has a single column. Then the optimal controller that minimizes the H_2-norm of the transfer function from w to z is the controller in Figure 5.3, where Φ_p has the following impulse response:*

$$\Phi_p(t) = [\mathbb{1}(t) - \mathbb{1}(t - h)]e^{-A_p^T(t-h)}M. \tag{5.27}$$

Here M is the stabilizing solution of the Riccati equation (5.6)*, while $A_p = A - MB_2B_2^T$. Notice that Φ_p has a finite impulse response with support on $[0, h]$.*

Proof. It may be verified that the controller in Figure 5.3 generates the optimal u given by (5.26) when driven by $w(t + h) = \delta(t)$. □

The optimal H_2-norm

The squared optimal H_2-norm is equal to the optimal cost function (5.9), which is given in the following theorem.

Theorem 5.5 (Single input optimal H_2-norm). *Consider the control system of Figure 5.1(b) where the plant's dynamics are governed by* (5.1)*. Suppose that Assumptions A1, A2, and A3 are satisfied and that w is a one-dimensional signal, i.e. $B_1 = b_1$ has a single column. Let M be the stabilizing solution of the Riccati equation* (5.6)*, while $A_p = A - MB_2B_2^T$. Furthermore, let X be the solution of the Lyapunov equation:*

$$A_pX + XA_p^T + B_2B_2^T = 0. \tag{5.28}$$

Then the squared optimal H_2-norm of the transfer function from w to z is

$$J_{opt}(h) = b_1^T M b_1 - b_1^T M (X - e^{A_p h} X e^{A_p^T h}) M b_1. \tag{5.29}$$

Proof. It follows from (5.16,5.17) that

$$\frac{d}{dt}(p^T x) = p^T \dot{x} + x^T \dot{p} = u_{1,\text{opt}}^T u_{1,\text{opt}} + x^T Q x. \tag{5.30}$$

Taking the integral of both sides of (5.30), the optimal value of the integral term in (5.9) is

$$\int_0^h \left(x^T Q x + u_{1,\text{opt}}^T u_{1,\text{opt}} \right) dt = p(h)^T x(h) - p(0)^T x(0)$$
$$= p(h)^T x(h). \tag{5.31}$$

The expression of $p(t)$ is readily available in (5.25), while the expression for $x(t)$ may be computed from the differential equation (5.20) and the initial condition (5.21,5.23). It is given by

$$x(t) = \Sigma_{11}(t) x(0) + \Sigma_{12}(t) q(0)$$
$$= -\Sigma_{12}(t) e^{A_p^T h} M b_1, \tag{5.32}$$

where

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} = e^{St}, \tag{5.33}$$

with

$$S = \begin{bmatrix} A_p & B_2 B_2^T \\ 0 & -A_p^T \end{bmatrix}. \tag{5.34}$$

Plugging (5.31) into (5.9) and using (5.25,5.32) to simplify the expression, the following is obtained:

$$\min_{u_1} \left((x(h) + b_1)^T M(x(h) + b_1) + \int_0^h \left(x^T Q x + u_1^T u_1 \right) dt \right)$$
$$= b_1^T M b_1 - b_1^T M \Sigma_{12}(h) e^{A_p^T h} M b_1, \tag{5.35}$$

where M is the solution of the Riccati equation (5.6), $\Sigma(t)$ is given by (5.33), and $A_p = A - B_2 B_2^T M$. The formula (5.35) may be further simplified by finding a simpler expression for Σ_{12}. By defining

$$W = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}, \tag{5.36}$$

where X is the solution of the Lyapunov equation (5.28), it is straightforward to compute

$$W S W^{-1} = \begin{bmatrix} A_p & 0 \\ 0 & -A_p^T \end{bmatrix}. \tag{5.37}$$

Note that since A_p is Hurwitz, the Lyapunov equation (5.28) has a unique solution. Using (5.37), it may be shown that

$$
e^{St} = \begin{bmatrix} e^{A_p t} & Xe^{-A_p^T t} - e^{A_p t}X \\ 0 & e^{-A_p^T t} \end{bmatrix}, \tag{5.38}
$$

implying that

$$
\Sigma_{12}(t) = Xe^{-A_p^T t} - e^{A_p t}X. \tag{5.39}
$$

Plugging (5.39) into (5.35) results in a simplified expression of the optimal cost which is equal to the squared optimal H_2 norm:

$$
J_{\text{opt}}(h) = b_1^T M b_1 - b_1^T M (X - e^{A_p h} X e^{A_p^T h}) M b_1. \tag{5.40}
$$

$\square$

Compared to the formula given in [Koj04], which involves solving a differential Riccati equation, the formula (5.29) appears simpler and only requires solving the Lyapunov equation (5.28) and computing the exponential of a matrix.

Effect of the preview time h on the H_2 performance

The first derivative of the optimal squared H_2-norm with respect to h may be computed as follows

$$
\begin{aligned}
\frac{\partial J_{\text{opt}}(h)}{\partial h} &= b_1^T M \frac{\partial}{\partial h}(e^{A_p h} X e^{A_p^T h}) M b_1 \\
&= b_1^T M (e^{A_p h} A_p X e^{A_p^T h} + e^{A_p h} X A_p^T e^{A_p^T h}) M b_1 \\
&= b_1^T M e^{A_p h}(A_p X + X A_p^T) e^{A_p^T h} M b_1 \\
&= -b_1^T M e^{A_p h} B_2 B_2^T e^{A_p^T h} M b_1 \le 0
\end{aligned} \tag{5.41}
$$

Evidently, the squared optimal H_2-norm as a function of h is non-increasing. Thus, as the preview time increases, the performance increases as well. Moreover, the first term in the right hand side of (5.29) is the optimal squared H_2-norm for $h = 0$ (i.e. no preview), so that the second term may be viewed as the performance gain owing to the previewed input. The minimum achievable H_2-norm is obtained if we set $h = \infty$ (i.e. infinite preview), which gives

$$
J_{\text{opt}, h=\infty} = b_1^T (M - MXM) b_1. \tag{5.42}
$$

5.4. Multiple inputs case

The optimal controller for the multiple inputs case is a straightforward extension of the single-input result. It turns out that the optimization problem may be reduced to N (the dimension of the external input w) separate single-input optimizations, to which the results from the single input case apply. Furthermore, the resulting controller has exactly the same structure as in the single-input case.

Corollary 5.6 (Multiple inputs solution). *Consider the control system of Figure 5.1(b) where the plant's dynamics are governed by* (5.1)*. Suppose that Assumptions A1, A2, and A3 are satisfied. The optimal controller that minimizes the H_2-norm of the transfer function from w to z is the controller in Figure 5.3. Here M is the stabilizing solution of the Riccati equation* (5.6)*, while the impulse response of Φ_p is given by* (5.27)*. Moreover the squared optimal H_2-norm is given by*

$$\mathrm{tr}(B_1^T M B_1 - B_1^T M (X - e^{A_p h} X e^{A_p^T h}) M B_1), \qquad (5.43)$$

where $A_p = A - B_2 B_2^T M$, and X is the solution of the Lyapunov equation (5.28)*.*

Proof. Recall (see Section 1.3.3) that the squared H_2-norm of the control system of Figure 5.1(b) may be computed by conducting N experiments, with N the dimension of w, as described in what follows. For the k-th experiment, the k-th element of w is set to the delayed delta function $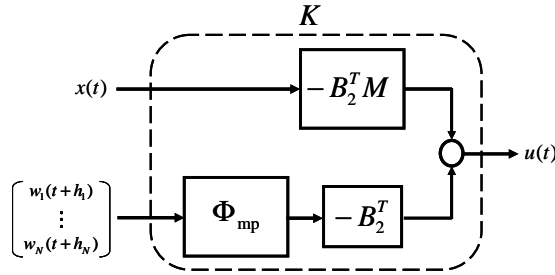\delta(t-h)$ while the other elements are set to zero. The squared H_2-norm of the closed loop system is then obtained by summing up the squared L_2-norm of the output z for all N experiments.

Since for each experiment only one element of w is active, for each experiment the control system may be recast into one with scalar external input w. Hence, the single-input results of the previous section applies. Using Theorem 5.4, it is straightforward to prove that for each experiment, the controller of Figure 5.3 generates the optimal input. This implies that the controller is the optimal controller. The squared optimal H_2-norm is obtained by summing up the optimal cost for all N experiments, which individually may be computed using (5.29). □

5.5. Multiple preview times case

In this section we treat the general case where each component of the exogenous input $w(t)$ that is fed to controller may have different preview times. The equivalent multiple preview times setup in which $w(t)$ is delayed before being fed to the plant P is shown in Figure 5.4. As in the single preview time case, the dynamics of the plant $P(s)$ are governed by (5.1). The difference is that instead of a single delay operator e^{-sh}, here we have a multiple
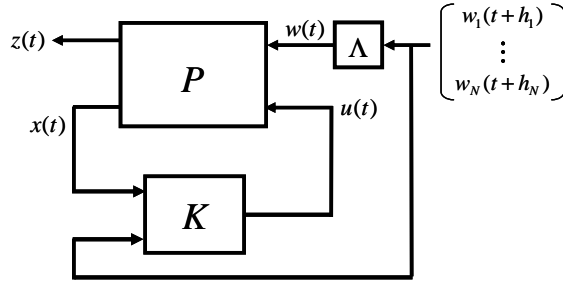
Figure 5.4.: The equivalent setup for the multiple preview times case.

delay operator $\Lambda(s)$ of the form:

$$\Lambda(s) = \operatorname{diag}(e^{-sh_1}, \ldots, e^{-sh_N}), \tag{5.44}$$

where N is the dimension of w and $h_k \geq 0, k = 1, \ldots, N$.

It turns out that the arguments on which the proof of Corollary 5.6 is based may also be applied. The results for the multiple preview time case is summarized in the following corollary.

Corollary 5.7 (Multiple preview times solution). *Consider the control system of Figure 5.4 where the plant's dynamics are governed by* (5.1) *and the delay operator $\Lambda(s)$ is given by* (5.44)*. Suppose that Assumptions A1, A2, and A3 are satisfied. The optimal controller that minimizes the H_2-norm of the transfer function from w to z is the controller in Figure 5.5. There, the block Φ_{mp} is a system with finite impulse response with the k-th column of its impulse response, denoted by $\Phi_{mp,k}(t)$, is given by*

$$\Phi_{mp,k}(t) = e^{-A_p^T(t-h_k)} M B_{1,k}(\mathbb{1}(t) - \mathbb{1}(t-h_k)) \tag{5.45}$$

where M is the solution of the Riccati equation (5.6)*, $A_p = A - B_2 B_2^T M$, and $B_{1,k}$ denotes the k-th column of the matrix B_1. Moreover the squared optimal H_2-norm is given by*

$$\operatorname{tr} B_1^T M B_1 - \sum_{k=1}^{N} (B_{1,k}^T M (X - e^{A_p h_k} X e^{A_p^T h_k}) M B_{1,k}), \tag{5.46}$$

where X is the solution of the Lyapunov equation (5.28)*.*

Proof. As in the proof of Corollary 5.6, the squared H_2-norm of the control system of Figure 5.4 may be computed by conducting N experiments, as described in what follows. For the k-th experiment, the k-th element of w is set to the delayed delta function $\delta(t - h_k)$ while the other elements are set

Figure 5.5.: The optimal controller, multiple preview times case.

to zero. The squared H_2-norm of the closed loop system is then obtained by summing up the squared L_2-norm of the output z for all N experiments. As in the multiple inputs single preview time case, for each experiment the problem also reduces to a single-input problem. The only difference is that here the preview time is different for each experiment. Nevertheless, the results from Section 5.3 still apply for each experiment. Applying the single input results to each experiment related to a particular component of w will result in a particular column of the optimal controller.

Using Theorem 5.4, it is straightforward to ascertain that the controller of Figure 5.5, where the FIR block Φ_{mp} is described by (5.45), generates the optimal input for each experiment. The squared optimal H_2 norm is obtained by summing up the optimal cost of the N experiments, resulting in the expression:

$$\sum_{k=1}^{N}(B_{1,k}^T M B_{1,k} - B_{1,k}^T M(X - e^{A_p h_k} X e^{A_p^T h_k}) M B_{1,k}), \qquad (5.47)$$

which is equal to the expression (5.46). $\qquad\qquad\square$

5.6. Numerical example

In this section we present an example of the multiple preview times case.

Consider the multiple preview times setup of Figure 5.4. Let the plant P be governed by the state equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_2 u(t),\end{aligned} \qquad (5.48)$$

where

$$A = 1, \ B_1 = \begin{bmatrix} b_{1,1} & b_{1,2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \ B_2 = \sqrt{3}$$
$$C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.49}$$

It may be verified that Assumptions A1, A2, and A3 are satisfied. Furthermore, the multiple delay operator $\Lambda(s)$ is given by

$$\Lambda(s) = \begin{bmatrix} e^{-sh_1} & 0 \\ 0 & e^{-sh_2} \end{bmatrix}, \ h_1, h_2 \geq 0. \tag{5.50}$$

Notice that w_1 and w_2 may have different preview times.

The optimal controller and the optimal H_2 norm may be computed using Corollary 5.7. The first step is to compute the stabilizing solution M of the Riccati equation (5.6), which in this case becomes a quadratic equation

$$1 + 2M - 3M^2 = 0. \tag{5.51}$$

The stabilizing solution of the above Riccati equation is

$$M = 1. \tag{5.52}$$

Next we compute the matrix A_p:

$$A_p = A - B_2 B_2^T M = -2. \tag{5.53}$$

The optimal controller is shown in Figure 5.5, where the impulse response of the FIR block Φ_{mp} may be computed using (5.45):

$$\Phi_{mp}(t) = \begin{bmatrix} \Phi_{mp,1}(t) & \Phi_{mp,2}(t) \end{bmatrix}, \tag{5.54}$$

where

$$\Phi_{mp,1}(t) = e^{2(t-h_1)}(\mathbb{1}(t) - \mathbb{1}(t - h_1)),$$
$$\Phi_{mp,2}(t) = 2e^{2(t-h_2)}(\mathbb{1}(t) - \mathbb{1}(t - h_2)). \tag{5.55}$$

To compute the optimal H_2 norm we need to compute the solution X of the Lyapunov equation (5.28) which in this case is

$$X = \frac{3}{4}. \tag{5.56}$$

The squared optimal H_2 norm can then be computed using (5.46), which results in the expression

$$5 - \frac{3}{4}(1 - e^{-4h_1}) - 3(1 - e^{-4h_2}). \tag{5.57}$$
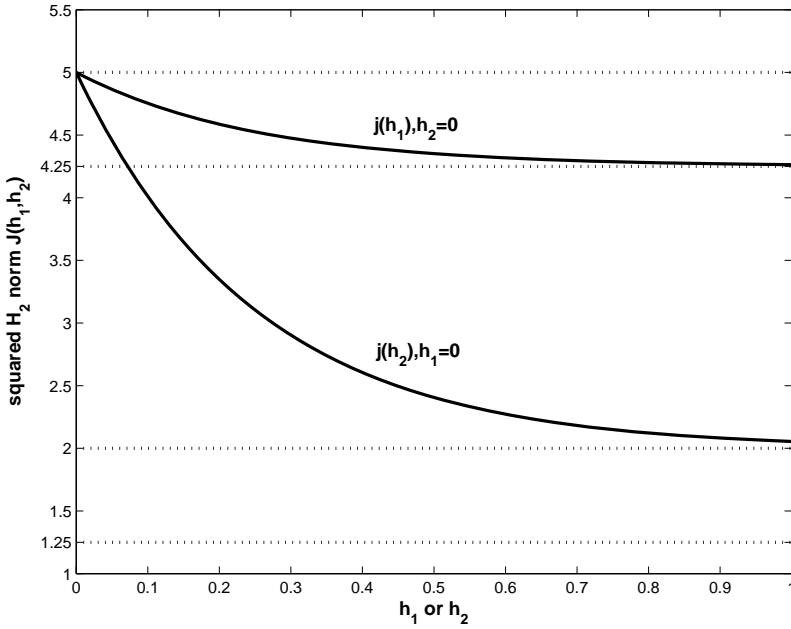
Figure 5.6.: The squared optimal H_2-norm as a function of the preview times.

The first term of (5.57) is the squared optimal H_2 norm for zero preview times, while the second and the third terms, i.e. $-\frac{3}{4}(1 - e^{-4h_1})$ and $-3(1 - e^{-4h_2})$, are the performance gain owing to the preview times h_1 and h_2, respectively. The maximum achievable squared H_2 norm corresponding to $h_1, h_2 = \infty$ is 1.25. Since the expression (5.57) is affine in h_1 and h_2, we may display the plots of the performance gain owing to each preview time in a single one-dimensional plot. This plot is shown in Figure 5.6.

Notice that increasing the preview times beyond around 1 time unit only gives very small performance gain. This can be explained by observing the general expression (5.46). There, the exponential term $e^{A_p h_k}$ is practically zero if

$$h_k \gg \frac{1}{|\operatorname{Re} \lambda_{\max}(A_p)|},$$

where $\lambda_{\max}(A_p)$ denotes the eigenvalue of A_p with the the largest real part in absolute value. This analysis is similar to the result in [And69] in the context of the smoothing problem, where it is shown that setting the smoothing lag to approximately five times the dominant time constant results in a performance gain that is very close to the one corresponding to infinite smoothing lag.
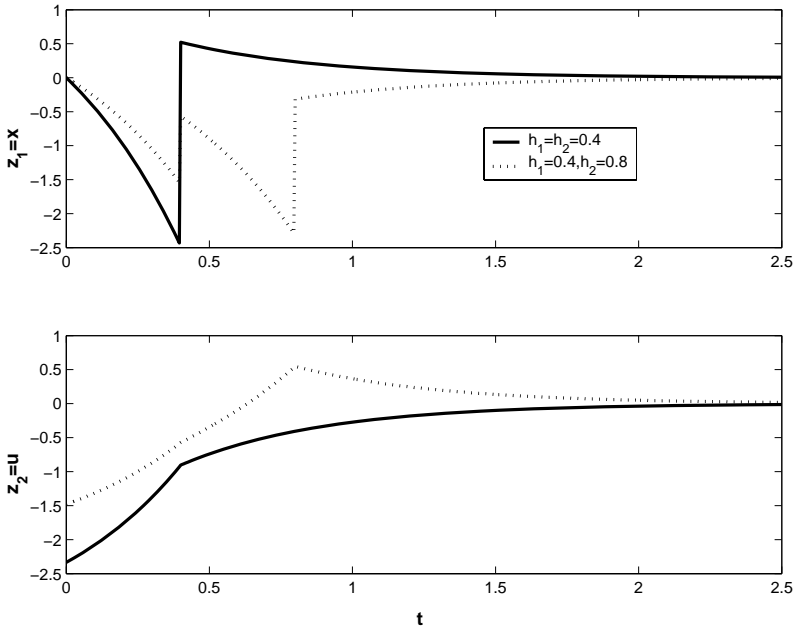
Figure 5.7.: The output z, top: $z_1 = x$, bottom: $z_2 = u$.

In our example, $|\operatorname{Re} \lambda_{\max}(A_p)| = 2$, so that increasing the preview times beyond several times of $\frac{1}{2}$ is practically useless. Figure 5.6 also shows that the plot corresponding to h_2 is steeper than the one corresponding to h_1. This suggests that increasing the preview time of the second channel is more advantageous than increasing the one of the first channel. Thus, it makes sense to use different preview times for different channels.

Now suppose we drive the system with an external input of the form

$$w(t) = \begin{bmatrix} \delta(t - h_1) \\ \delta(t - h_2) \end{bmatrix}.$$

The response

$$z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

is shown in Figure 5.7 for two values of the pair (h_1, h_2). One plot is for the case where both channels are previewed with the same preview time ($h_1 = h_2 = 0.4$) while the other plot is for the case where the channels are previewed differently ($h_1 = 0.4, h_2 = 0.8$).

5.7. Relaxing assumption A3

Assumption A3 allows us to formulate the LQR problem (5.2,5.3). The assumption may be relaxed using the well-known method of input substitution. The method works by introducing the state feedback

$$u(t) = R^{-\frac{1}{2}}v(t) - R^{-1}D_2^T C_1 x(t) \tag{5.58}$$

in (5.1), where $R = D_2^T D_2$ and v is the new input. With this change of the input, the state equation becomes

$$\dot{x}(t) = \bar{A}x(t) + b_1\delta(t - h) + \bar{B}_2 v(t), \ x(0) = x_0, \tag{5.59}$$

while the cost criterion is given by

$$\min_v \int_0^\infty x(t)^T \bar{Q}x(t) + v(t)^T v(t)dt \tag{5.60}$$

where $\bar{Q} = C_1^T(I - D_2 R^{-1}D_2^T)C$, $\bar{A} = (A - B_2 R^{-1}D_2^T C_1)$, and $\bar{B}_2 = B_2 R^{-\frac{1}{2}}$. The resulting LQR problem (5.59,5.60) is of the same form as the problem (5.2,5.3) and therefore may be solved using the technique from the previous sections.

5.8. Concluding remarks

In this chapter, the H_2 control problem of preview systems is considered. The single input case is first solved. Based on the single input results, the multiple inputs case and the multiple preview times case are treated. The results show that by providing the external input in advance to the controller, the H_2 performance of the control system may be improved. This chapter also provides a formula for the optimal H_2-norm that clearly shows how the performance gain owing to the previewed input varies as the preview time increases.

6

Case Studies

It is not the aim of this chapter to provide a complete control design demonstration of a real world control problem. It has a more modest objective of illustrating some potential of the theories developed in the previous chapters for application.

Therefore, the emphasis is not on how to design an H_2 controller for a certain plant according to certain requirements. Rather, the emphasis is on how the formulas derived in the previous chapters may practically be employed to compute the solution of practical H_2-optimization problems and to simulate the resulting control system.

Two case studies are presented: one involves a system with output delays (a rolling mill) and the other deals with a preview system (a tracking problem involving a container crane). In addition, the important problem of approximating systems with finite impulse response (FIR systems), which always appears as a part of the controllers derived in the preceding chapters, is also discussed.

6.1. Steel rolling mill (a delay system)

In [GH98], a control problem concerning a steel rolling mill that involves multiple output delays is presented. There, the control problem is posed as an LQG optimization problem. However, the solution developed in [GH98] is limited in the sense that it assumes that each disturbance signal affects only one measured output, ensuring that the delay operator commutes with the plant. Since the rolling mill control problem does not satisfy this assumption, changes to the model had to be made, which resulted in a controller that is not only non-optimal but also of high state dimension.

In this section we revisit the LQG problem introduced in [GH98], convert it to an equivalent H_2 control problem and obtain the true optimal controller for the unmodified model (in contrast with the non-optimal one based on a modified model obtained in [GH98]) using the time-domain technique of Chapter 3. The rolling mill plant and the associated control
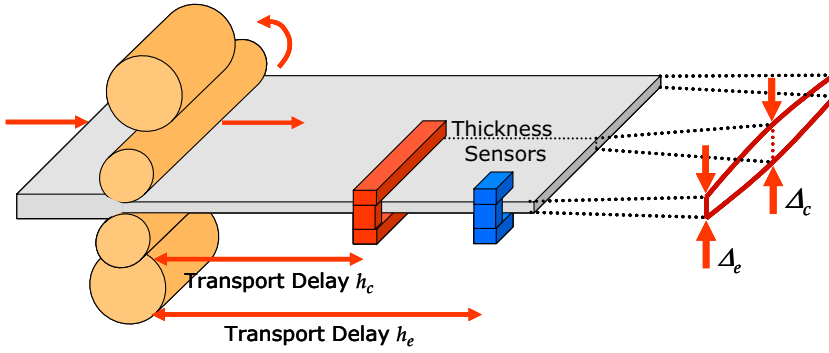
Figure 6.1.: The steel rolling mill.
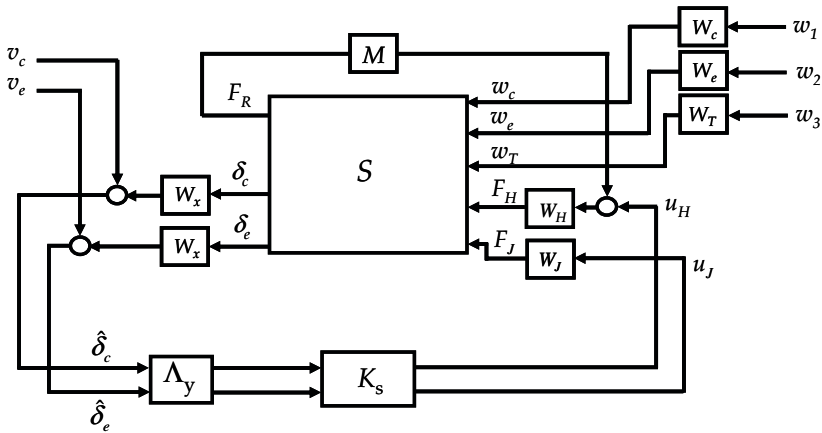
Figure 6.2.: The rolling mill thickness regulation model.

problem is elaborated in what follows. The description of the plant and its model is based on the paper [GH98].

Steel rolling mill

A steel rolling mill consists of a series of stands, which is designed to gradually reduce the thickness of a steel sheet. Typically, a rolling mill takes in a steel sheet of 30 mm thickness at 1000ºC and reduces the thickness gradually over several stands to 2 mm at 800ºC.

The stands consist of rollers, between which the steel sheet is pressed. As the rollers rotate, the steel sheet is drawn into the gap between the rollers and pressed so that its thickness is reduced. At each stand there is a local thickness regulation that utilizes the measured force applied to the rollers. However, it turns out that local thickness regulation does not achieve zero steady-state error.

Another problem is that the incoming steel sheet entering a stand may not be uniform. There may be fluctuations in both the thickness and the temperature. These fluctuations in the incoming steel sheet result in thickness fluctuations in the outgoing steel sheet. To remove the steady state error and suppress the thickness and temperature disturbances, an additional loop involving X-ray thickness sensors is installed at the last stand, as depicted in Figure 6.1.

In this section, we only discuss the regulation problem concerning the last stand. The shape of the gap at the last stand determines the cross section shape of the output steel sheet. The cross section profile of the outgoing steel sheet is monitored by taking two measurements:

- the center line thickness Δ_c

- the edge thickness Δ_e.

Due to certain circumstances, the X-ray sensors have to be placed at a distance from the rollers, resulting in measurement delays. In this case, two X-ray sensors are employed to measure the center line thickness Δ_c and the edge thickness Δ_e. Since the two sensors are placed at different distances from the rollers, the measurement delays of Δ_c and Δ_e are different. Here the sensor for the center line thickness is placed closer to the rollers than the sensor for the edge thickness so that the delay affecting Δ_c is smaller than the one affecting Δ_e.

To control the shape of the gap between the rollers, two actuators are used at the last stand:

- the hydraulic capsules, which apply a force to the rollers adjusting their vertical position

- the bending jacks, which bend the rollers to form the desired shape of the gap between the rollers.

Linearized model

The linearized mathematical model of the thickness regulator control system of the last stand of the rolling mill is shown in Figure 6.2. The stand itself is modeled as a static system S. Its inputs are the thickness and temperature disturbances (w_c, w_e, w_T) along with the pressing force F_H produced by the hydraulic capsules and the bending force F_J produced by the bending jacks. Its outputs are the roller force F_R, the centerline thickness error signal δ_c, and the edge thickness error signal δ_e. Mathematically, the input-output relation for the stand S is given by

$$\begin{bmatrix} F_R \\ \delta_c \\ \delta_e \end{bmatrix} = S \begin{bmatrix} w_c \\ w_e \\ w_T \\ F_H \\ F_J \end{bmatrix}. \tag{6.1}$$

One of the outputs of the stand, the roller force F_R is fed back through the static feedback gain M to the hydraulic capsules. The static gain M is given and will not be part of our controller design problem.

The centerline thickness disturbance w_c, the edge thickness disturbance w_e, and the temperature disturbance w_T represent the thickness and temperature fluctuations in the incoming steel sheet. They are modeled as filtered white noise:

$$w_c = \underbrace{\frac{C_c}{\tau_c s + 1}}_{W_c(s)} w_1, \quad w_e = \underbrace{\frac{C_e}{\tau_e s + 1}}_{W_e(s)} w_2, \quad w_T = \underbrace{\frac{C_T}{\tau_T s + 1}}_{W_T(s)} w_3, \tag{6.2}$$

where w_1, w_2, and w_3 are zero mean unity variance white noise.

The X-ray sensors are also modeled as first order systems of the form:

$$W_x(s) = \frac{C_x}{\tau_x s + 1}. \tag{6.3}$$

In the linearized model, the output of the sensors are the thickness error signals (δ_c, δ_e), which represent the deviation of the actual thickness signals (Δ_c, Δ_e) from desired values. The X-ray sensors measurements are corrupted by zero mean Gaussian white noise v_c and v_e having variances σ_c^2 and σ_e^2, respectively. The noisy measurements are denoted by $\hat{\delta}_c$ and $\hat{\delta}_e$, which are delayed by the output delay operator

$$\Lambda_y = \mathrm{diag}(e^{-sh_c}, e^{-sh_e}), \quad h_e > h_c. \tag{6.4}$$

The delayed noisy measurements are then fed to the controller K_s. In turn, the controller outputs the control signals u_H and u_J that drive the hydraulic capsules W_H and the bending jacks W_J, respectively. The actuators are given by

$$W_H(s) = \frac{C_H}{\tau_H s + 1}, \quad W_J(s) = \frac{C_J}{\tau_J s + 1}. \tag{6.5}$$

All the numerical values of the mill model may be found in the following list.

- Stand model:

$$S = \begin{bmatrix} 7.05 \times 10^7 & 1.35 \times 10^9 & -3.15 \times 10^3 & -2.34 \times 10^9 & 1.50 \times 10^{-1} \\ 6.40 \times 10^{-1} & 9.20 \times 10^{-2} & -8.41 \times 10^{-7} & 3.74 \times 10^{-1} & 2.63 \times 10^{-11} \\ 3.10 \times 10^{-2} & 7.17 \times 10^{-1} & -8.66 \times 10^{-7} & 3.56 \times 10^{-1} & 2.17 \times 10^{-11} \end{bmatrix}$$

- Disturbances weights

$$W_c(s) = W_e(s) = \frac{10^{-3}}{0.1s + 1}, \quad W_T(s) = \frac{10}{0.1s + 1}$$

- Local thickness regulation gain matrix

$$M = 2.58 \times 10^{-10}$$

- Actuators

$$W_H(s) = \frac{1}{0.03s + 1}, \quad W_J(s) = \frac{10^{10}}{0.03s + 1}$$

- X-ray sensors

$$W_x(s) = \frac{1}{0.01s + 1}$$

- Measurement delays

$$h_c = 0.1\text{s}, \quad h_e = 0.25\text{s}$$

- Measurement noise variances

$$\sigma_c^2 = \sigma_e^2 = 10^{-11}.$$

Controller design

To convert the control system of Figure 6.2 to the output delay standard configuration of Figure 3.1(b) on page 32, we identify the external signal w, the control input u, and the measured output y from Figure 6.2:

$$w = [w_1, w_2, w_3, \hat{v}_c, \hat{v}_e]^T, \quad u = [u_H, u_J]^T, \quad y = [\hat{\delta}_c, \hat{\delta}_e]^T. \qquad (6.6)$$

Here, $\hat{v}_c$ and $\hat{v}_e$ are the normalized version of v_c and v_e having unity variance. The controlled output signal z does not appear in the model of Figure 6.2. This signal reflects the control objectives and may be recovered from the performance index of the LQG formulation in [GH98].

We convert the performance index of the LQG problem in [GH98] to its equivalent expression of the output signal z in the standard configuration of Figure 3.1(b). Like in [GH98], to achieve constant disturbance rejection, it is then multiplied with an approximate integrator:

$$z = \frac{1}{s + \epsilon} \tilde{z},$$
$$\tilde{z} = \begin{bmatrix} Q^{\frac{1}{2}}[x_c, x_e]^T \\ R^{\frac{1}{2}}[u_H, u_J]^T \end{bmatrix}. \qquad (6.7)$$

Here Q and R are weighting matrices given by

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = \begin{bmatrix} 10^{-3} & 0 \\ 0 & 10^{-3} \end{bmatrix}$$

while the quantities x_c and x_e are the states of the two X-ray sensors corresponding to δ_c and δ_e, respectively. The constant ϵ is a small positive number which is the pole of the approximate integrator $\frac{1}{s+\epsilon}$. Here, we take

$$\epsilon = 0.001.$$

The choice of the signal $\tilde{z}$ in (6.7) is to make the thickness error $y = [\delta_c, \delta_e]^T$ small while keeping the control input reasonably small. In this case the weights Q and R regulate the trade off between the two objectives. Multiplying the signal with an approximate integrator aims to force the controller to include an integration action, which results in a desirable property of constant disturbance rejection.

Having identified the signals w, y, and u of the output delays standard configuration in Figure 3.1(b) from the model Figure 6.2 and designed the output signal z according to the requirements, it is straightforward to construct the generalized plant P of the standard configuration for the rolling mill problem. The objective now is to find the controller K_{s} such that the closed loop system is internally stable and

$$\|F_\ell(P(s), K_{\mathrm{s}}(s)\Lambda_y(s))\|_2 \qquad (6.8)$$
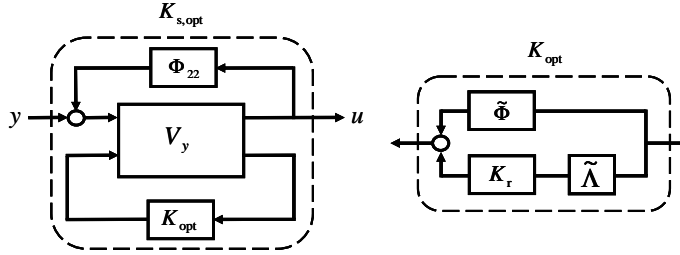
Figure 6.3.: Optimal controller structure for the steel rolling mill.

is minimized.

The next step is apply Lemma 3.3 on page 36 to the standard problem (6.8), converting it to a simplified problem with the plant's (1,2) and (2,1) blocks invertible. The lemma stipulates that minimizing (6.8) is equivalent to minimizing

$$\|F_\ell(G(s), K_s(s)\Lambda_y(s))\|_2 \tag{6.9}$$

for a certain plant G whose (1,2) and (2,1) blocks are biproper. Converting the simplified problem (6.9) to a regulator problem of the form of Figure 3.2 is accomplished by applying Lemma 3.6 on page 40. It is shown that

$$F_\ell(G(s), K_s(s)\Lambda_y) = G_{11}(s) + K(s)\Lambda_y(s)G_{21}(s) \tag{6.10}$$

in which we have a proper bijection between K and K_s. Taking the transpose of the right hand side of (6.10), which does not change the H_2-norm, we obtain the regulator problem, the solution of which has been developed in Section 3.4 and Section 3.5.

We follow the above procedure to obtain the optimal controller. The optimal controller K_s structure is depicted in Figure 6.3. We can see that that the controller consists of two rational blocks (V_y and K_r), two FIR blocks (Φ_{22} and $\tilde{\Phi}$), and a multiple delay operator $\tilde{\Lambda} = \text{diag}(e^{-s(h_e-h_c)}, 1)$. The state dimension of the two rational blocks are equal to the state dimension of the plant P.

Simulation Results

As in [GH98], we simulate the response of the closed loop system to constant[1] temperature and thickness disturbances w_c, w_e, w_T in the incoming steel sheet. Figure 6.4 shows the thickness error response for 0.5 mm step

---

[1]Although the thickness and the temperature disturbances are modeled as white noise as commonly done in the LQG setting, in the deterministic H_2 setting these disturbances are equivalent to unit impulses. Since integrators are employed as weights in the controlled output z, it is expected that the control system will reject constant (step) disturbances.
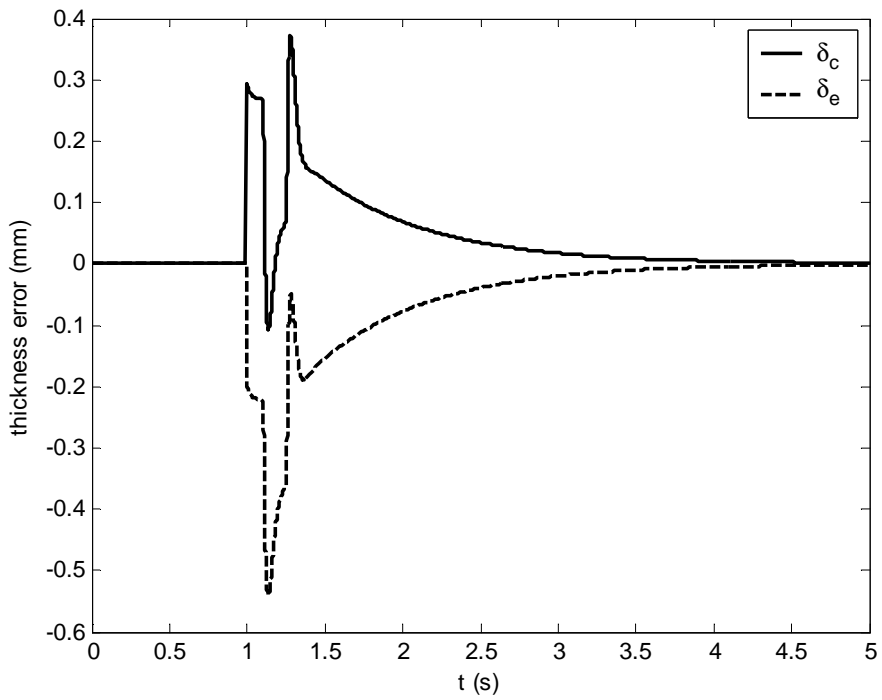
Figure 6.4.: Thickness error response for step thickness disturbances.
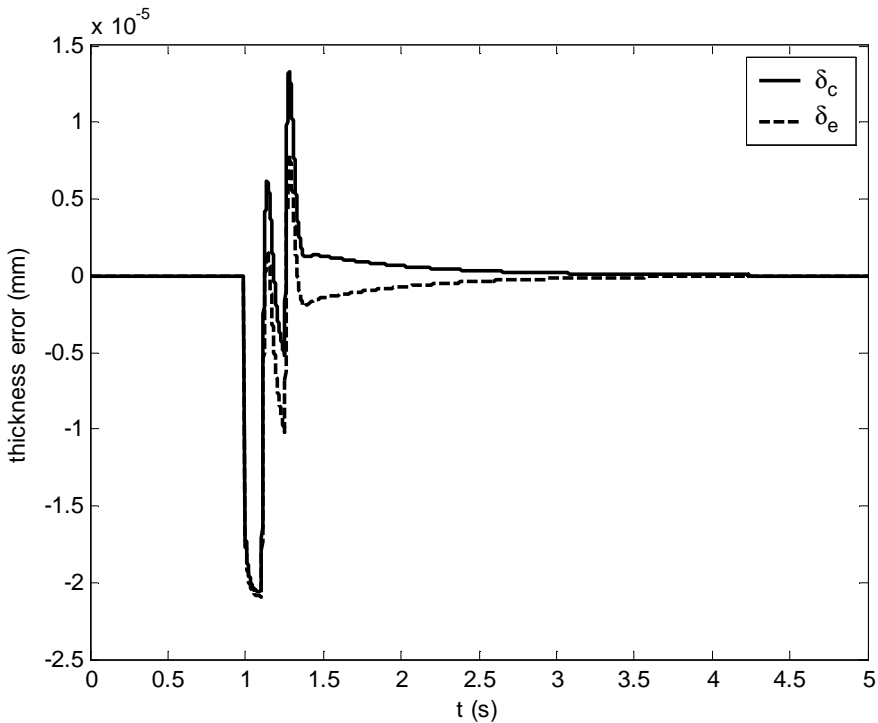
Figure 6.5.: Thickness error response for step temperature disturbance of 20º.

center line thickness disturbance at $t = 1$ and -0.3 mm step edge thickness disturbance at $t = 1$ in the incoming steel sheet, i.e.

$$w_c(t) = 0.5\mathbb{1}(t-1), \quad w_e(t) = -0.3\mathbb{1}(t-1), \quad w_T(t) = 0. \tag{6.11}$$

We observe that the thickness errors observed in the outgoing steel sheet quickly go to zero. Similarly, Figure 6.5 shows the thickness error response for a step temperature disturbance of 20º at $t = 1$ in the incoming steel sheet, i.e.

$$w_c(t) = w_e(t) = 0, \quad w_T(t) = 20\mathbb{1}(t-1). \tag{6.12}$$

Again, we see that the thickness error in the outgoing steel sheet goes to zero. The simulation indicates that the controller achieves stability and constant disturbance rejection.

Implementation Problem

In the simulation, a problem was encountered. It is evident that for sufficiently large delay values, the simulated control system, which is theoretically stable, is practically unstable. The problem was traced and it turns out that the problem was caused by round-off error. For large delay values, the impulse response of the Φ_{22} block in the optimal controller (see Figure 6.3) becomes very large in magnitude. This results in the output signal of Φ_{22} becoming very large. When it is added to the much smaller signal y, the resulting signal suffers from round-off error, which somehow destabilizes the closed loop system.

6.2. Container crane (a preview system)

A container crane is a crane that is used to load cargo containers to a container ship at harbors. A two dimensional model of a container crane is shown in Figure 6.6. The container crane basically consists of a car with mass m_{car} that moves on a horizontal rail. The rail is suspended at certain height above the ground and covers the distance between the location of the container at the harbor and the destination ship, which is denoted by E. The crane works by connecting cables, which in our simple model is represented by a single cable with length L, from the car to the container. The container, which has mass m_{con}, is then moved to the ship by exerting the force F onto the car. The position of the car relative to its starting position onshore is denoted by x_{car}, while the angle of the cable as depicted in Figure 6.6 is denoted by θ. The position of the container relative to the car is denoted by x_{con} and is given by $x_{\text{con}} = x_{\text{car}} \sin\theta$. This quantity measures the lateral swinging motion of the container as the car moves along the rail.

The objective is to determine the force $F(t)$ that allows the container to move from the shore to the ship reasonably quickly but without causing the container to swing too much.

A nonlinear mathematical model of the container crane system based on the two dimensional model of Figure 6.6 may be derived from first principles. The model is given by the nonlinear state equation

$$
\dot{x}_{\text{c}} = \begin{bmatrix} \dot{x}_{\text{c},1} \\ \dot{x}_{\text{c},2} \\ \dot{x}_{\text{c},3} \\ \dot{x}_{\text{c},4} \end{bmatrix} = \begin{bmatrix} x_{\text{c},2} \\ \frac{F - m_{\text{con}} L x_{\text{c},4}^2 \sin(x_{\text{c},3}) - m_{\text{con}} g \sin(x_{\text{c},3}) \cos(x_{\text{c},3})}{m_{\text{car}} + m_{\text{con}} \sin^2(x_{\text{c},3})} \\ x_{\text{c},4} \\ \frac{F \cos(x_{\text{c},3}) - m_{\text{con}} L x_{\text{c},4}^2 \sin(x_{\text{c},3}) \cos(x_{\text{c},3}) - (m_{\text{car}} + m_{\text{con}}) g \sin(x_{\text{c},3})}{L(m_{\text{car}} + m_{\text{con}} \sin^2(x_{\text{c},3}))} \end{bmatrix}
\tag{6.13}
$$

where g is the gravitational constant and $x_{\text{c}}(t)$ is the state of system given
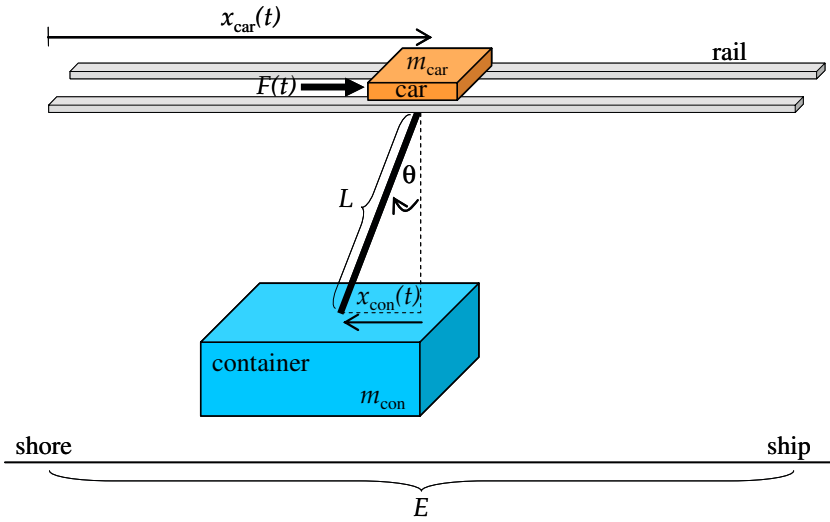
Figure 6.6.: A two dimensional model of a container crane.

by

$$x_c(t) = \begin{bmatrix} x_{c,1}(t) \\ x_{c,2}(t) \\ x_{c,3}(t) \\ x_{c,4}(t) \end{bmatrix} = \begin{bmatrix} x_{car}(t) \\ \dot{x}_{car}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix}. \tag{6.14}$$

A linearized model around the equilibrium $x_c = 0, F = 0$ may be obtained:

$$\dot{x}_c(t) = A_c x_c(t) + B_c u(t), \quad u(t) = F(t), \tag{6.15}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m_{con} g}{m_{car}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(m_{car} + m_{con}) g}{m_{con} L} & 0 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0 \\ \frac{1}{m_{car}} \\ 0 \\ \frac{1}{m_{car} L} \end{bmatrix}$$

Arbitrary numerical values of the container crane model are listed below.

- Car mass: $m_{car} = 100 \,\text{kg}$

(a)

(b)
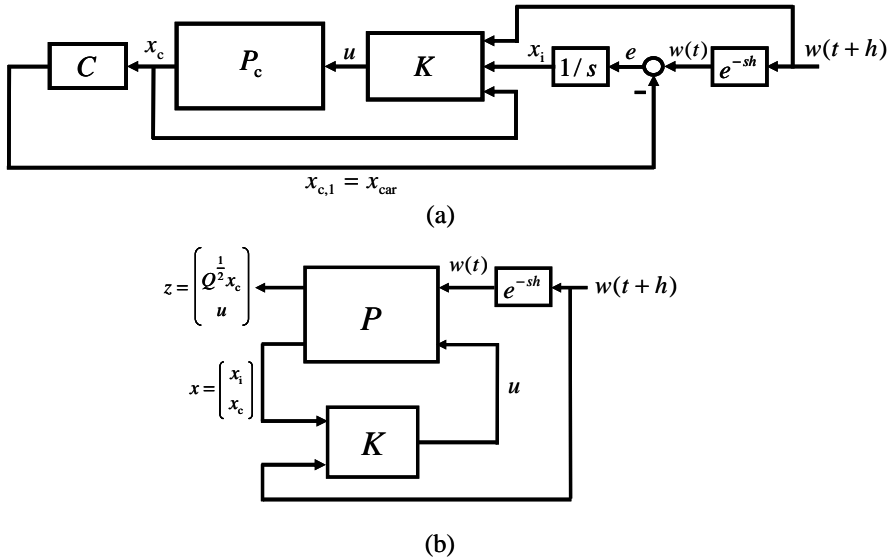
Figure 6.7.: The container crane control system as a tracking problem with preview (a), and its equivalent standard configuration (b).

- Container mass: $m_{\mathrm{con}} = 1000\,\mathrm{kg}$

- Cable length: $L = 10\,\mathrm{m}$

- Gravitational constant: $g = 9.82\,\mathrm{m/s^2}$

- Distance from shore to ship: $E = 25\,\mathrm{m}$.

The container crane as a tracking problem with preview

The control problem associated with the container crane may be formulated as a tracking problem with preview. The idea is that the car should follow a pre-determined reference position trajectory, which is denoted by $w(t)$. Since the reference is known in advance, we may use the preview technique developed in Chapter 5 to achieve better performance.

The configuration of the container crane control system as a tracking problem with preview is shown in Figure 6.7(a). There, P_{c} is the container crane model interconnected with a controller K, while the matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

is used to select the first component of the state x_{c}. In synthesizing the controller, we use the linearized model (6.15) of the container crane plant P_{c}, while for the simulation we shall use the nonlinear model (6.13).

Observe that the controller is fed with the time-advanced reference trajectory $w(t+h)$, while the position error signal $e(t)$ is formed by taking the difference between the non-advanced reference $w(t)$ and the actual position $x_{c,1}(t) = x_{car}$. Before being fed to the controller, the position error $e(t)$ is passed through an integrator. Note that the resulting integrated position error signal $x_i(t)$ is also the state of the integrator.

It is straightforward to recast the configuration Figure 6.7(a) to the standard configuration of Figure 6.7(b) used in Chapter 5. The equivalent standard configuration is shown in Figure 6.7(b). The generalized plant P is governed by the state equation:

$$\dot{x} = Ax + b_1 w(t) + B_2 u(t), \tag{6.16}$$

where

$$x = \begin{bmatrix} x_i \\ x_c \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & -C \\ 0 & A_c \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{m_{con}g}{m_{car}} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{(m_{car}+m_{con})g}{m_{con}L} & 0 \end{bmatrix},$$

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ B_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_{car}} \\ 0 \\ \frac{1}{m_{car}L} \end{bmatrix}.$$

Notice that the state of the generalized plant consists of the state of the container crane plant P_c augmented with the state of the integrator.

The controlled output signal $z(t)$ in Figure 6.7(b) reflects the objective of the control system. In this case we choose

$$z(t) = \begin{bmatrix} Q^{\frac{1}{2}} x(t) \\ u(t) \end{bmatrix}, \tag{6.17}$$

where Q is given by

$$Q = \text{diag}(150^2, 0, 0, 500^2, 500^2).$$

The idea is to penalize the integrated position error x_i to give good tracking, while keeping the swinging motion of the container small by penalizing θ and $\dot{\theta}$.

Now that we have converted the problem to the standard preview configuration, we may use results from Chapter 5, to compute the optimal H_2-controller.
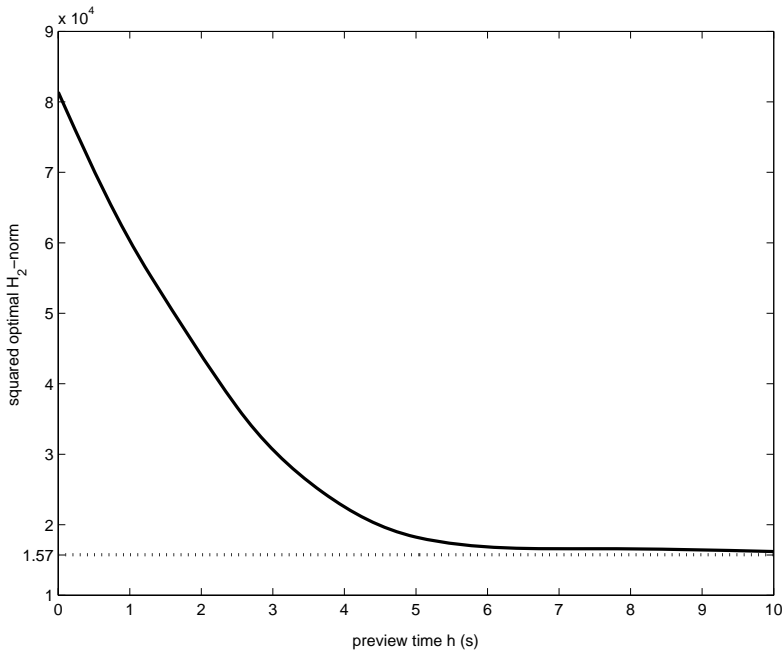
Figure 6.8.: The squared optimal H_2-norm of the container crane tracking system as a function of the preview time.

The first step is to determine the preview time h to be used in the tracking system. This is done by plotting the squared optimal H_2-norm as a function of the preview time[2] (Figure 6.8). We can see that increasing the preview time above around $h = 6$ only gives minor performance gain. Therefore, we select

$$h = 6$$

as our preview time.

The controller can then be computed using Theorem 5.4 on page 97. The controller consists of a static state feedback and a FIR block, as depicted in Figure 5.3 on page 97.

Before we can simulate the container crane tracking system, we have to select the reference trajectory that the car will follow. It is (more or less arbitrarily) chosen to be of the following form:

$$w(t + h) = \begin{cases} 25 \sin(\frac{2\pi}{60}t), & 0 \leq t \leq 15 \\ 25, & t > 15 \end{cases} \tag{6.18}$$

---

[2]This may be computed using Theorem 5.5 on page 97.
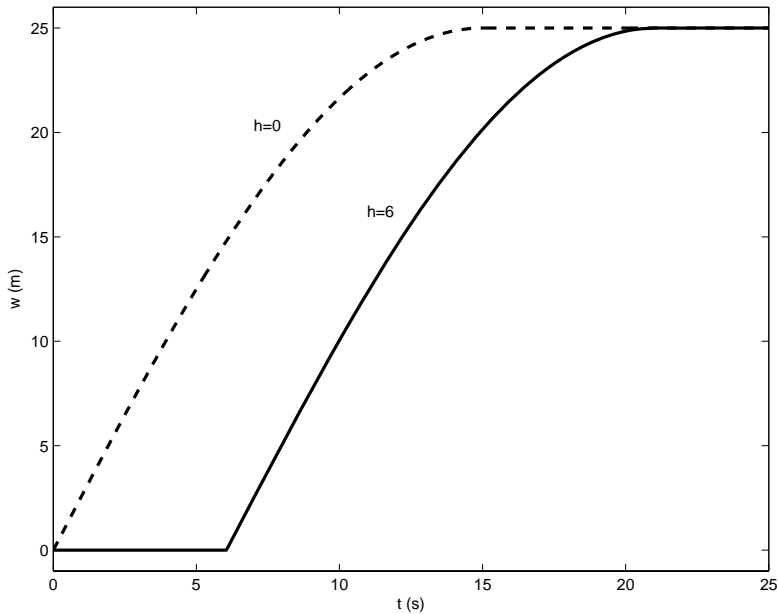
Figure 6.9.: The car reference trajectory for $h = 0$ and $h = 6$.

which means that after being idle for h seconds, the car position follows a sinusoidal function (amplitude 25 m, period 60 s) for a quarter of the period (15 s) until it reach the 25 m mark and then the car stops. The reference trajectory $w(t)$ is shown in Figure 6.9 for $h = 0$ and $h = 6$.

Simulation results

Having obtained the optimal controller and chosen the reference trajectory, we are ready to conduct the simulation. The simulation is carried out using the configuration of Figure 6.7(a). Here, we use the non-linear model of the container crane system, given by (6.13). We use a preview time equal to 6s, as selected earlier. For a benchmark, we also simulate without using preview ($h = 0$). Figure 6.10 shows the car position and velocity as functions of time, while Figure 6.11 shows the input force exerted to the car and the container lateral swinging motion relative to the position of the car as functions of time.

Compared to the optimal non-preview controller, the optimal controller with preview clearly exhibits better performance. Although in the preview case, the reference trajectory is delayed by 6s, in both cases (preview and non-preview), it takes around 25s for the car to settle at its destination at the 25m mark. Hence, in term of the amount of time required to transport the
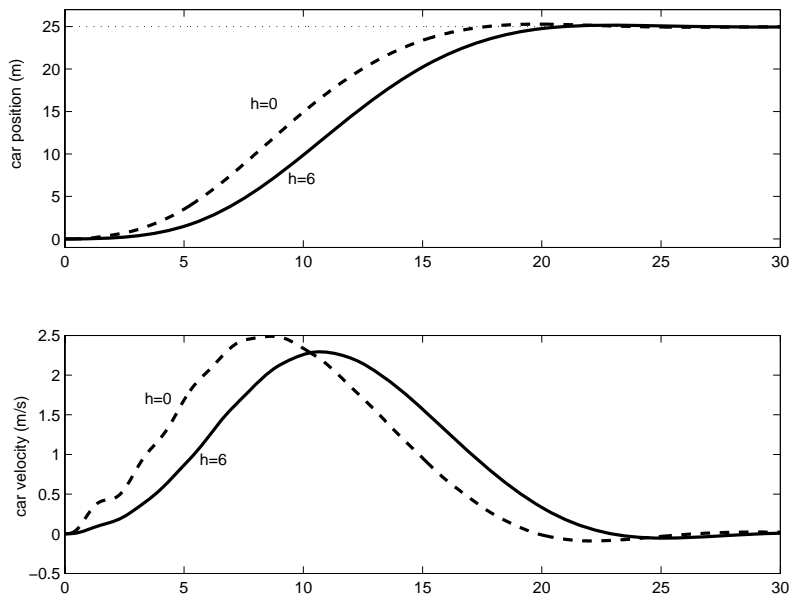
Figure 6.10.: The car position $x_{\mathrm{car}}(t)$ (top) and velocity $\dot{x}_{\mathrm{car}}(t)$ (bottom).
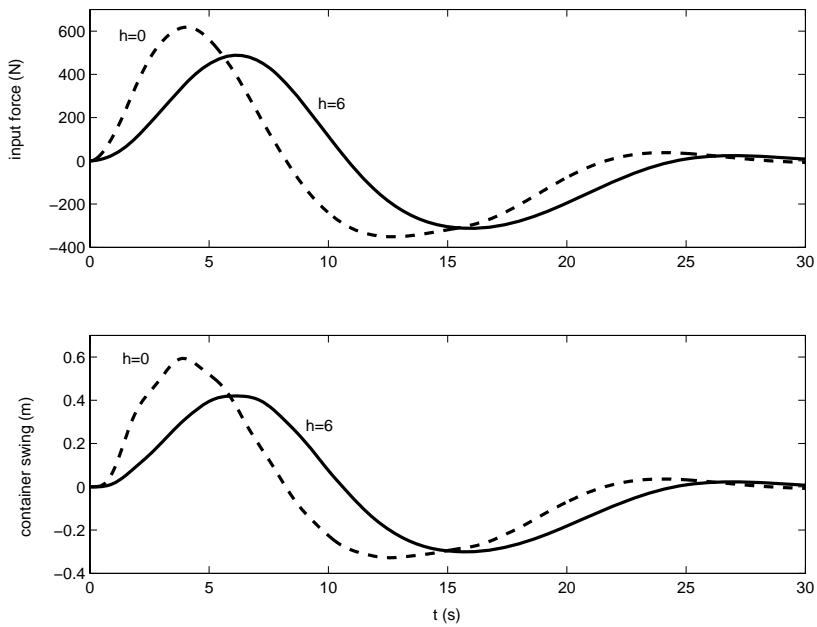
Figure 6.11.: The input force $F(t)$ (top) and the container position relative to the car position $x_{\mathrm{con}}(t)$ (bottom).

container, both controllers have comparable performance. Nevertheless, the preview controller seems to outperform the non-preview controller in other areas. Apart from the fact that control system with preview seems to use less force (see Figure 6.11), the amplitude of the container lateral swinging motion is also smaller in the control system with preview. Overall, it is safe to say that the preview system provides better performance than the non-preview system. The following table shows a numerical comparison of the performance of the two controllers.

Property	$h = 6$ s	$h = 0$ s
maximum car position overshoot	16 cm	25 cm
maximum car velocity	2.3 m/s	2.5 m/s
maximum input force	490 N	620 N
maximum container lateral swing	42 cm	59 cm

6.3. Approximation of FIR systems

It is evident that FIR systems seem to appear in every controller for systems with i/o delays and preview systems derived in this thesis. Therefore, their practical implementation is an important issue to discuss. In this section, a method to implement the FIR systems by representing them as a discrete state space systems is presented. It is a slight extension of the method in [Kui03].

Finite impulse response blocks found in this thesis in general may be represented as an impulse response matrix, the components of which are SISO impulse response functions having different, but finite, support. Each component may be written as the completion of a rational transfer function

$$
\begin{aligned}
\phi(s) &= \pi_h \left(e^{-sh} F(s) \right) = \pi_h \left(e^{-sh} \left[\begin{array}{c|c} A_F & b_F \\ \hline c_F & 0 \end{array} \right] \right) \\
&= \left[\begin{array}{c|c} A_F & b_F \\ \hline c_F e^{-A_F h} & 0 \end{array} \right] - e^{-sh} \left[\begin{array}{c|c} A_F & b_F \\ \hline c_F & 0 \end{array} \right].
\end{aligned}
\tag{6.19}
$$

If $F(s)$ is stable, then $\phi(s)$ may be implemented using (6.19) as the difference between two rational stable transfer functions, one of which is delayed. However, if F is not stable, another method has to be used. The impulse response of ϕ is given by

$$
\begin{aligned}
\phi(t) &= c_F e^{-A_F h} e^{A_F t} b_F \mathbb{1}(t) - c_F e^{A_F(t-h)} b_F \mathbb{1}(t-h) \\
&= c_F e^{A_F(t-h)} b_F (\mathbb{1}(t) - \mathbb{1}(t-h)),
\end{aligned}
\tag{6.20}
$$

which clearly shows that ϕ has a finite impulse response with support on $[0, h]$. Using the impulse response (6.20), the response $y(t)$ of ϕ to an input

$u(t)$ may be expressed as a convolution, which in this case is a finite integral:

$$y(t) = \phi(t) * u(t) = \int_0^h \phi(\tau)u(t-\tau)d\tau. \tag{6.21}$$

The finite integral (6.21) may for instance be approximated by a finite sum using the trapezoidal rule:

$$y(t) \approx t_s \left(\frac{1}{2}\phi(0)u(t) + \sum_{j=1}^{N-1} \phi(jt_s)u(t-jt_s) + \frac{1}{2}\phi(h)u(t-h) \right) \tag{6.22}$$

where N is the number of samples and

$$t_s = \frac{h}{N}.$$

The approximation (6.22) may be implemented as a discrete time system with sampling time t_s. The discrete time system is described by the following state equation:

$$\begin{aligned} x(k+1) &= A_{\mathrm{d}}x(k) + B_{\mathrm{d}}u(k) \\ y(k) &= C_{\mathrm{d}}x(k) + D_{\mathrm{d}}u(k), \end{aligned} \tag{6.23}$$

where

$$A_{\mathrm{d}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{6.24}$$

$$B_{\mathrm{d}} = \mathrm{col}[0, 0, \dots, 0, 1] \tag{6.25}$$

$$C_{\mathrm{d}} = \begin{bmatrix} \frac{1}{2}t_s\phi(h) & t_s\phi((N-1)t_s) & \dots & t_s\phi(t_s) \end{bmatrix}, \tag{6.26}$$

$$D_{\mathrm{d}} = \frac{1}{2}t_s\phi(0). \tag{6.27}$$

Here the state x stores the last N values of the input:

$$x(k) = \mathrm{col}[u(k-N), u(k-N+1), \dots, u(k-1)].$$

The discrete time system (6.23), when complemented by zero order hold blocks for discrete/continous conversion, may be used in implementing FIR systems.

The implementation of FIR systems using (6.23) may easily be extended to represent MIMO systems with finite impulse response. It may also be

extended to FIR MIMO systems associated with systems with multiple delays, with an impulse response of the form

$$\Phi(t) = \begin{bmatrix} \Phi_{11}(t)(\mathbb{1}(t) - \mathbb{1}(t - h_{11})) & \dots & \Phi_{1p}(t)(\mathbb{1}(t) - \mathbb{1}(t - h_{1p})) \\ & \ddots & \\ \Phi_{m1}(t)(\mathbb{1}(t) - \mathbb{1}(t - h_{m1})) & \dots & \Phi_{mp}(t)(\mathbb{1}(t) - \mathbb{1}(t - h_{mp})) \end{bmatrix}, \tag{6.28}$$

provided that we can find a sample time t_s, with which we can express

$$h_{ij} = N_{ij}t_s, \quad i = 1, \dots, m, \ j = 1, \dots, p$$

where N_{ij}s are non-negative integers. In this case the state needs to store past values of all components of the input, and is given by

$$x(k) = \text{col}[u_1(k - N_{\max,1}), \dots, u_1(k - 1), \dots, u_p(k - N_{\max,p}), \dots, u_p(k - 1)],$$

where $N_{\max,j}$ is the maximum value of N_{ij} for the j-th column associated with the j-th component of the input:

$$N_{\max,j} = \max(N_{1j}, \dots, N_{mj}).$$

The discrete time system that approximate (6.23) is given by

$$x(k+1) = \begin{bmatrix} A_{\mathrm{d},1} & & & 0 \\ & A_{\mathrm{d},2} & & \\ & & \ddots & \\ 0 & & & A_{\mathrm{d},p} \end{bmatrix} x(k) + \begin{bmatrix} B_{\mathrm{d},1} & & & 0 \\ & B_{\mathrm{d},2} & & \\ & & \ddots & \\ 0 & & & B_{\mathrm{d},p} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} C_{\mathrm{d},11} & C_{\mathrm{d},12} & \dots & C_{\mathrm{d},1p} \\ \vdots & \vdots & & \vdots \\ C_{\mathrm{d},m1} & C_{\mathrm{d},m2} & \dots & C_{\mathrm{d},mp} \end{bmatrix} x(k) + \begin{bmatrix} D_{\mathrm{d},11} & D_{\mathrm{d},12} & \dots & D_{\mathrm{d},1p} \\ \vdots & \vdots & & \vdots \\ D_{\mathrm{d},m1} & D_{\mathrm{d},m2} & \dots & D_{\mathrm{d},mp} \end{bmatrix} u(k), \tag{6.29}$$

where the blocks $A_{\mathrm{d},j}$ and $B_{\mathrm{d},j}$, $j = 1, \dots, p$, have the following dimensions

$$A_{\mathrm{d},j} \in \mathbb{R}^{N_{\max,j} \times N_{\max,j}}$$
$$B_{\mathrm{d},j} \in \mathbb{R}^{N_{\max,j} \times 1}$$

and posses the structure exactly like the matrix A_{d} and B_{d}, respectively, in (6.24,6.25). The blocks $C_{\mathrm{d},ij}$ and $D_{\mathrm{d},ij}$, where $i = 1, \dots, m$ and $j = 1, \dots, p$, are given by

$$C_{\mathrm{d},ij} = \begin{bmatrix} 0 & \dots & 0 & \frac{1}{2}\phi_{ij}(N_{ij}t_s) & \phi_{ij}((N_{ij} - 1)t_s) & \dots & \phi_{ij}(t_s) \end{bmatrix},$$
$$D_{\mathrm{d},ij} = \frac{1}{2}\phi_{ij}(0),$$

with $C_{\mathrm{d},ij} \in \mathbb{R}^{1 \times N_{\max,j}}$.

Stability concern

In the paper [vADLR99], the question regarding the stability of the approximation of FIR blocks using methods similar to (6.22) is raised. It is demonstrated by numerical examples that certain approximations of the FIR block may cause instability in the closed loop system. Although the problem does not surface in the case studies described in the preceding sections, stability analysis (see for example [Mir04] or [Zho04]) shows that the approximation (6.22) indeed may destabilize the closed loop system even when the sample time t_s approaches zero.

In [Mir04], the root of the problem is identified as a combination of poor high-frequency accuracy of the approximation method and excessive sensitivity to high-frequency additive plant uncertainties. To mitigate the problem, one of the method suggested in [Mir04] is to make a simple modification to the implementation of the FIR blocks. There, it is shown that

$$\phi(s) = \pi_h \left(e^{-sh} F(s) \right)$$
$$= \frac{\tau s + 1}{\tau s + 1} \pi_h \left(e^{-sh} F(s) \right)$$
$$= \frac{1}{\tau s + 1} \left((\tau s + 1) \left(\left[\begin{array}{c|c} A_F & b_F \\ \hline c_F e^{-A_F h} & 0 \end{array} \right] - e^{-sh} \left[\begin{array}{c|c} A_F & b_F \\ \hline c_F & 0 \end{array} \right] \right) \right)$$
$$= \frac{1}{\tau s + 1} \left(\tau c_F e^{-A_F h} b_F - \tau c_F b_F e^{-sh} + \pi_h \left(\left[\begin{array}{c|c} A_F & b_F \\ \hline c_F(I + \tau A_F) & 0 \end{array} \right] e^{-sh} \right) \right)$$
$$= \frac{1}{\tau s + 1} \tilde{\phi}(s),$$

where $\tau > 0$ but otherwise arbitrary. It is also shown that by approximating the FIR block $\tilde{\phi}(s)$, instead of $\phi(s)$, the instability problem may be averted. See [Mir04] for more details.

6.4. Concluding Remarks

In this chapter, the potential of the theory derived in this thesis for application is demonstrated. It is shown that it is feasible to compute the optimal controller for real world problems and to simulate the resulting control system.

6. Case Studies

7

Conclusions and Future Research

This chapters outlines the main contributions of this thesis. It also offers a number of open problems for future research.

7.1. Conclusions

The main theme of this thesis is the standard H_2 problem of systems with multiple input/output delays. The research effort in solving the problem resulted in two approaches to the problem: the frequency domain approach and the time domain approach. Both have their own strong and weak points.

The time domain approach, which is reported in Chapter 3, offers straightforward formulation of the solution to the problem. The approach makes use of dynamic programming ideas, adapted for the delay system, which results in simple formulas and controller structure. The resulting controller consists of blocks with rational transfer function and blocks with finite impulse response, both of which are implementable. Moreover, the state dimension of the rational part of the controller is equal to the state dimension of the plant. The only obvious drawback is that, at least in its current form, the method only works for the case where the delays are present solely on one side of the controller, either in the input or the output channels.

The frequency domain approach, which is reported in Chapter 4, is more general in the sense that it can handle systems with delays in both the input and the output channels. The approach, which is based on spectral factorization arguments, also produces an optimal controller consisting of rational blocks and finite impulse response blocks. However, the formulas are more complicated and less elegant than their time domain approach counterparts. Another drawback is that the method cannot handle unstable plants.

The development of the two solutions gives rise to the solution of two closely related problems, namely the parametrization of all stabilizing controllers for systems with multiple i/o delays and the H_2 control problem of preview systems.

The Youla-Kučera parametrization for systems with multiple i/o delays, which is reported in Chapter 2, is a generalization of the single delay result from Mirkin and Raskin [MR03]. It is clearly a contribution in its own right, but it also plays an important role in the frequency domain solution. It is a necessary preliminary result for reducing the standard H_2 problem to the equivalent two-sided regulator problem.

It turns out that the ideas from Chapter 3 may also be applied to solve H_2 control problem for preview systems. This results in Chapter 5.

It is important to note that the results of this thesis clearly possess the potential for application. Chapter 6 demonstrates this fact by presenting two case studies that apply the theory from this thesis to real control problems.

7.2. Future research

The following text lists several open problems that may be explored for future research.

Extension of the H_2 time domain solution. The time domain solution of the H_2 control problem of systems with multiple i/o delays in Chapter 3 only covers the case where the delays occur only on one side of the controller. The extension to the case where delays occur on both sides of the controller is still an open problem.

H_2 control problem of combined delay/preview systems. This problem is a combination of the delay problem and the preview problem. Combined delay/preview systems are systems where delays occur in the control input u and the controller is supplied with a time-advanced version of the external input w. In H_∞ setting, the problem is treated in [KI03a]. The solution of the problem in the H_2 setting is still open and may be achieved by exploiting and combining previously obtained results in delay systems and preview systems.

More applications. As stated earlier, the results from this thesis possess the potential for application. Since delays commonly occur in many physical systems, there should be many real world problems that can be explored. One interesting example is congestion control in telecommunication networks, in particular the internet. In recent years, we have been witnessing the rise of internet applications that requires certain degree of quality of service, such as voice over IP and streaming video. These demanding applications make network performance a very important subject. One of the key issues affecting network performance is congestion control. It turns out that the mechanism of congestion control in the internet may be modeled as a feedback control system with delays (see for example [HMTG02]). Therefore, it is possible to apply results from control theory, in particular

the results from this thesis, to this problem.

7. Conclusions and Future Research

A

Results from Optimal Control Theory

A.1. Principle of optimality

The following theorem states the fundamental principle in optimal control, namely the principle of optimality (see for example [AM89] for details).

Theorem A.1 (Principle of optimality). *Consider the system*

$$\dot{x}(t) = f\left(x(t), u(t)\right), \qquad x(t_0) = x_0. \tag{A.1}$$

The following criterion function is associated with the system:

$$J_{[t_0, t_e]}(x_0, u) = g\left(x(t_e)\right) + \int_{t_0}^{t_e} f_0\left(x(t), u(t)\right) dt. \tag{A.2}$$

If $u_{opt}(\cdot)$ is an optimal input for the system (A.1) *with the cost criterion* (A.2) *with initial condition $x(0) = x_0$, then the restriction of u_{opt} to the interval $[t_1, t_e]$ is an optimal control for the system* (A.1) *with initial condition $x(t_1) = x_{opt}(t_1)$ and with cost criterion*

$$J_{[t_1, t_e]}(x_0, u) = g\left(x(t_e)\right) + \int_{t_1}^{t_e} f_0\left(x(t), u(t)\right) dt. \tag{A.3}$$

Furthermore, we have the following properties:

$$\min_u J_{[t_0, t_e]}(x_0, u) = \int_{t_0}^{t_1} f_0\left(x_{opt}(t), u_{opt}(t)\right) dt + \min_u J_{[t_1, t_e]}\left(x_{opt}(t_1), u\right); \tag{A.4}$$

$$J_{[t_0, t_e]}(x_0, u_{opt}) = \int_{t_0}^{t_1} f_0\left(x_{opt}(t), u_{opt}(t)\right) dt + J_{[t_1, t_e]}\left(x_{opt}(t_1), u_{opt}\right). \tag{A.5}$$

A.2. Pontryagin minimum principle

This section outlines the Pontryagin minimum principle that may be used to solve optimal control problems. The text is taken from [AM89].

Theorem A.2 (Pontryagin minimum principle). *Consider the system*

$$\dot{x} = f(x, u, t) \tag{A.6}$$

and performance index

$$V(x(0), u(\bullet)) = \int_0^T l(x(\tau), u(\tau), \tau)d\tau + m(x(T)). \tag{A.7}$$

Suppose that f, l, $\partial f/\partial x$, and $\partial l/\partial x$ satisfy the Lipschitz condition. Define, with p termed the costate vector,

$$H(x, u, t, p) = p^T f(x, u, t) + l(x, u, t) \tag{A.8}$$

and

$$H_{opt}(x, t, p) = \min_{u=u(x,t,p)} H(x, u, t, p) \tag{A.9}$$

(assuming the minimum exists and at the minimum, $\frac{\partial H}{\partial u} = 0$). Then the equations

$$\dot{x} = \frac{\partial H}{\partial p}, \quad x(0) \text{ prescribed}$$
$$\dot{p} = -\frac{\partial H_{opt}}{\partial x}, \quad p(T) = \left.\frac{\partial m}{\partial x}\right|_{x(T)} \tag{A.10}$$

are satisfied along the optimal trajectory, and if $x_{opt}(\bullet), p_{opt}(\bullet)$ denote the solution of (A.10) *corresponding to the optimal trajectory, the optimal $u_{opt}(\bullet)$ is*

$$u_{opt}(t) = \arg\min H(x_{opt}(t), u, t, p_{opt}(t)). \tag{A.11}$$

A.3. Linear quadratic regulator problem

In this section we review the solution of the linear quadratic regulator problem, both for the finite horizon case and the infinite horizon case. The material given in this section is well-known. See for example the books [KS72] and [AM71].

A.3.1. Finite horizon LQR problem

Consider the following linear system equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$x(0) = x_0.$$

$$(A.12)$$

The regulator problem is the problem of finding an input that drives the state to zero fast, while keeping the input reasonably small. This objective is somewhat achieved by minimizing the following quadratic cost function (thus the name is linear quadratic regulator problem):

$$\int_0^T \|Cx(t) + Du(t)\|^2 dt + x(T)^T Mx(T).$$

$$(A.13)$$

with the assumption

$$R := D^T D > 0, \quad M \geq 0.$$

$$(A.14)$$

Linear quadratic regulator problem: Consider the system (A.12) with the assumption (A.14). Find the input $u(t), 0 \leq t \leq T$ such that (A.13) is minimized.

Removing the cross terms

To simplify the problem, we first remove the cross terms between the state and the input in the cost function by introducing the following pre state feedback:

$$u(t) = R^{-\frac{1}{2}} v(t) - R^{-1} D^T C x(t).$$

$$(A.15)$$

With this change of the input variable, the system equation becomes:

$$\dot{x} = (A - BR^{-1}D^T C)x + BR^{-\frac{1}{2}} v$$
$$x(0) = x_0$$

$$(A.16)$$

while the cost function is

$$\int_0^T (x^T Q x + v^T v) dt + x(T)^T Mx(T)$$

$$(A.17)$$

where

$$Q = C^T (I - DR^{-1}D^T)C \geq 0.$$

$$(A.18)$$

The solution

The following theorem gives the optimal input v_{opt} that minimizes (A.17). For a proof of the theorem we refer to [KS72].

Theorem A.3 (Finite horizon LQR). *Consider the LQR problem* (A.16) *with the performance criterion* (A.17)*. The following state differential equation:*

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} (A - BR^{-1}D^TC) & BR^{-1}B^T \\ Q & -(A - BR^{-1}D^TC)^T \end{bmatrix}}_{=:S} \begin{bmatrix} x \\ p \end{bmatrix} \tag{A.19}$$

$$x(0) = x_0, \qquad p(T) = -Mx(T)$$

has a unique solution and the optimal input is given by

$$v_{opt}(t) = R^{-\frac{1}{2}} B^T p(t). \tag{A.20}$$

Then, the input (A.20) *is the solution of the LQR problem (A.16,A.17).*

The optimal input for the original problem (A.12),(A.13) may be recovered using (A.15). However, first we need to compute the optimal state x and the co-state p trajectories by solving the differential equation (A.19). Define

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} = e^{St}, \tag{A.21}$$

where S is given by (A.19), then the state and co-state trajectories are given by

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Sigma(t) \begin{bmatrix} x(0) \\ p(0) \end{bmatrix}. \tag{A.22}$$

From (A.19), the initial state $x(0) = x_0$ is assumed to be known and $p(T) = -Mx(T)$. Thus by substituting these quantities into (A.22) after setting $t = T$, we may solve (A.22) to obtain $p(0)$, which is given by

$$p(0) = -\left(\Sigma_{22}(T) + M\Sigma_{12}(T)\right)^{-1}\left(\Sigma_{21}(T) + M\Sigma_{11}(T)\right)x_0$$
$$= C_\ell(\tilde{\Sigma}(T), M)x_0 \tag{A.23}$$

where

$$\tilde{\Sigma}(t) = \begin{bmatrix} \Sigma_{22}(t) & \Sigma_{21}(t) \\ -\Sigma_{12}(t) & -\Sigma_{11}(t) \end{bmatrix}. \tag{A.24}$$

Here $C_\ell(\tilde{\Sigma}, M) := -(\tilde{\Sigma}_{11} - M\tilde{\Sigma}_{21})^{-1}(\tilde{\Sigma}_{12} - M\tilde{\Sigma}_{22})$ denotes the left homographic linear fractional transformation. The full initial condition is then given by:

$$\begin{bmatrix} x(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} I \\ -\left(\Sigma_{22}(T) + M\Sigma_{12}(T)\right)^{-1}\left(\Sigma_{21}(T) + M\Sigma_{11}(T)\right) \end{bmatrix} x_0$$
$$= \begin{bmatrix} I \\ C_\ell(\tilde{\Sigma}(T), M) \end{bmatrix} x_0. \tag{A.25}$$

Thus, the optimal state and the optimal co-state trajectories may be computed:

$$x(t) = \Gamma(t)x_0$$
$$p(t) = \Xi(t)x_0,$$

(A.26)

where

$$\Gamma(t) = \Sigma_{11}(t) + \Sigma_{12}(t)C_\ell(\tilde{\Sigma}(T), M)$$
$$\Xi(t) = \Sigma_{21}(t) + \Sigma_{22}(t)C_\ell(\tilde{\Sigma}(T), M).$$

(A.27)

The optimal cost may then be computed by plugging in the optimal state and the optimal co-state to the criterion function. Alternatively we may derive a closed-form expression as follows. From (A.19) we have that

$$\frac{d}{dt}p^T x = p^T \dot{x} + x^T \dot{p}$$
$$= p^T \left((A - BR^{-1}D^T C)x + (BR^{-1}B^T)p \right) + x^T \left(Qx - (A - BR^{-1}D^T C)p \right)$$
$$= p^T BR^{-1}B^T p + x^T Qx$$
$$= v_{\text{opt}}^T v_{\text{opt}} + x^T Qx.$$

(A.28)

Hence by taking the integral of both sides of (A.28) and adding the final state penalty, we obtain:

$$x^T(T)Mx(T) + \int_0^T (x^T Qx + v_{\text{opt}}^T v_{\text{opt}})dt$$
$$= x^T(T)Mx(T) + p^T(T)x(T) - p^T(0)x(0)$$
$$= -p^T(0)x(0) = -x^T(0)\Xi^T(0)x(0).$$

(A.29)

The last two equations come from the fact that $p(t) = -Mx(T)$ and equation (A.27). Hence, the optimal cost is given by

$$\min_v \int_0^T (x^T Qx + v^T v)dt + x(T)^T Mx(T) = x_0^T M_0 x_0,$$

(A.30)

where

$$M_0 = -\Xi(0).$$

(A.31)

Moreover, if we observe the state-equation (A.19) and the optimal input expression given by (A.15, A.20), we may express the optimal input u_{opt} as a weighted sum of the state of the autonomous system having S as its A-matrix. The details are shown in the following lemma.

Lemma A.4 (Finite horizon LQR). *Consider the system represented by* (A.12). *The performance criterion is given by* (A.13) *with the assumption* (A.14). *Then the optimal input u_{opt} that minimizes the performance criterion is the output of the following autonomous system:*

$$\dot{z}(t) = \begin{bmatrix} (A - BR^{-1}D^T C) & BR^{-1}B^T \\ C^T(I - DR^{-1}D^T)C & -(A - BR^{-1}D^T C)^T \end{bmatrix} z(t),$$

$$z(0) = \begin{bmatrix} x_0 \\ -\left(\Sigma_{22}(T) + M\Sigma_{12}(T)\right)^{-1}\left(\Sigma_{21}(T) + M\Sigma_{11}^{-1}(T)\right) x_0 \end{bmatrix}, \quad \text{(A.32)}$$

$$u_{opt}(t) = \begin{bmatrix} -R^{-1}D^T C & R^{-1}B^T \end{bmatrix} z(t), \quad 0 \le t \le T.$$

A.3.2. Infinite horizon LQR problem

If in (A.13) we let T approach infinity and set M to zero, we obtain the infinite horizon LQR problem. The following theorem gives the well-known solution to the problem (see for example [AM71] or [KS72]).

Theorem A.5 (Infinite horizon LQR). *Consider the LQR problem* (A.12) *with the performance criterion* (A.17). *Assume*

- $R = D^T D > 0$

- (A, B) *is stabilizable and* (C, A) *is detectable*

- $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ *has full column rank for all ω.*

Let $T = \infty$ and $M = 0$. Let X be the unique stabilizing solution of the following algebraic Riccati equation:

$$Q + \hat{A}^T X + X\hat{A} - XBR^{-1}B^T X = 0 \quad \text{(A.33)}$$

where

$$\hat{A} = A - BR^{-1}D^T C, \quad Q = C^T(I - DR^{-1}D^T)C. \quad \text{(A.34)}$$

Then the optimal input that minimizes the performance criterion is

$$u_{opt} = Fx(t) \quad \text{(A.35)}$$

where

$$F = -R^{-1}\left(B^T X + D^T C\right). \quad \text{(A.36)}$$

Moreover, the optimal cost is given by $x_0^T X x_0$.

As in the finite horizon case, the optimal input may be modeled as the output of an autonomous system.

Corollary A.6 (Infinite horizon LQR). *The optimal input* (A.35) *may be expressed as the output of the following autonomous system:*

$$
\begin{aligned}
\dot{x}(t) &= (A + BF)\, x(t), \qquad x(0) = x_0 \\
u_{opt} &= Fx(t).
\end{aligned}
\tag{A.37}
$$

Furthermore, $A + BF$ is stable.

A. Results from Optimal Control Theory

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Summary

Time delay is a phenomenon that naturally occurs in the modeling of many physical systems. Examples of time delay systems include models in mechanics, physics, engineering, biology, medicine, and economy. Time delay may also function as a means of model reduction. High order and infinite dimensional systems may be well-approximated by a low order system with a delay. The presence of delays in a control system poses a challenge to controller design. Besides complicating system analysis and controller design, delays also make satisfactory control more difficult to achieve.

This thesis deals with control problems related to a class of delay systems where time delays are present in the input/output channels of a linear time invariant plant. The main theme concerns the standard H_2-optimal control problem of such systems. Two solutions are proposed: a time domain solution and a frequency domain one.

The time domain approach offers a straightforward formulation of the solution to the problem. The approach makes use of dynamic programming ideas adapted for the delay system, which results in simple formulas and controller structure. The resulting controller consists of blocks with rational transfer function and blocks with finite impulse response, both of which are implementable. Moreover, the state dimension of the rational part of the controller is equal to the state dimension of the plant. The only obvious drawback is that, at least in its current form, the method only works for the case where the delays are present solely on one side of the controller, either in the input or the output channels.

The frequency domain solution is more general in the sense that it can handle systems with delays in both the input and the output channels. The approach, which is based on spectral factorization arguments, also produces an optimal controller consisting of rational blocks and finite impulse response blocks. However, the formulas are more complicated and less elegant than their time domain approach counterparts. Another drawback is that the method cannot handle unstable plants.

The development of the two solutions gives rise to the solution of two closely related problems, namely the parametrization of all stabilizing controllers (Youla-Kučera parametrization) for systems with multiple i/o delays and the H_2 control problem of preview systems. The former provides a formulation that describes all controllers that internally stabilizes a given

system with i/o delays. It is not only a contribution in its own right, but also a necessary preliminary in the frequency domain solution of the standard H_2 problem. The latter problem concerns preview systems. These are systems where all or a part of the external input signal is known in advance. It turns out that ideas from the time domain solution of the standard H_2 problem for systems with i/o delays may also be adapted to solve the problem's preview system counterpart.

Samenvatting

In veel fysische systemen en hun modellen treedt het fenomeen tijdvertraging op. Voorbeelden van systemen met tijdvertraging zijn o.a. te vinden in de mechanica, fysica, elektrotechniek, biologie, geneeskunde en economie. Ook als in het systeem zelf geen vertraging voorkomt kan het soms handig zijn om in het model ervan wel vertragingen te introduceren als benadering van anders hoge-orde of oneindig-dimensionale componenten. De aanwezigheid van vertragingen maakt het regelprobleem complex. Het compliceert niet alleen de analyse en het regelontwerp van het systeem, het maakt het ook noodzakelijk de regeldoelen naar beneden bij te stellen.

Dit proefschrift behandelt regelproblemen voor een klasse van systemen waar de vertragingen enkel in de ingang en/of uitgang voorkomen. Het hoofdthema is H_2-optimale regeling voor dergelijke systemen. We beschouwen dit probleem in zowel tijddomein als frequentiedomein.

In het tijddomein laat het probleem zich makkelijk formuleren. Met behulp van dynamisch programmeren, toegespitst op systemen met vertragingen, worden eenvoudige formules en een duidelijke regelstructuur afgeleid. De regelaar kan worden opgevat als een interconnectie van een eindigdimensionaal systeem en een FIR-systeem. Hoewel de laatste oneindigdimensionaal is zijn beide deelsystemen, en daarmee de regelaar als geheel, goed te implementeren. De toestandsdimensie van het eindig-dimensionale deel is gelijk aan dat van het gegeven systeem. De in dit proefschrift gepresenteerde oplossing heeft als nadeel dat het enkel werkt voor het geval dat de vertragingen zich aan één kant van de regelaar bevinden, ofwel aan de ingang van de regelaar ofwel aan de uitgang van de regelaar.

De frequentiedomeinoplossing is meer algemeen toepasbaar in de zin dat het vertragingen toelaat aan tegelijk ingang en uitgang van de regelaar. Ook deze oplossing, gebaseerd op spectrale factorisatie, resulteert in een regelaar die gezien kan worden als interconnectie van een eindig-dimensionaal systeem en een FIR-systeem. De formules zijn echter wel gecompliceerder en minder transparant dan die van de tijddomeinoplossing. Een ander nadeel is dat de methode niet werkt voor instabiele gegeven systemen.

De ontwikkeling van de tijd- en frequentiedomeinoplossing geeft aanleiding tot twee gerelateerde problemen, te weten: de parameterisatie van alle stabiliserende regelaars (Youla-Kučera-parameterisatie) voor systemen met meerdere ingangs-uitgangsvertragingen en het H_2-optimale regelprobleem

voor systemen met voorkennis. De Youla-Kučera-parameterisatie beschrijft alle regelaars die een gegeven systeem met gegeven meerdere ingangs-uitgangsvertragingen intern stabiliseert. Dit resultaat is op zichzelf interessant, maar vormt ook een essentieel onderdeel van de frequentiedomeinoplossing van het H_2-probleem. Systemen met voorkennis zijn systemen waar het externe ingangssignaal deels of geheel een bepaalde tijd vooruit bekend is. Enkele idëen die zijn ontwikkeld voor de tijddomeinoplossing van het H_2-probleem voor systemen met meerdere vertragingen blijken ook van toepassing te zijn voor het probleem met voorkennis.

Ringkasan

Waktu tunda adalah sebuah fenomena yang banyak ditemukan dalam pe-modelan berbagai sistem. Sebagai contoh, waktu tunda dapat ditemui di model-model mekanika, fisika, rekayasa, biologi, kedokteran, dan ekonomi. Waktu tunda dapat pula digunakan sebagai alat reduksi model. Sistem dengan orde tinggi dan sistem dengan dimensi tak hingga seringkali dapat dihampiri dengan baik oleh sistem berorde rendah dengan waktu tunda. Keberadaan waktu tunda di suatu sistem kendali pada umumnya memper-sulit perancangan pengendali untuk sistem tersebut. Disamping itu, waktu tunda juga mempersulit analisis sistem dan berpotensi untuk memperbu-ruk kinerja sebuah sistem kendali.

Tesis ini membahas sebuah kelas dari sistem waktu tunda, yaitu sistem waktu tunda dimana hanya sinyal-sinyal masukan atau keluaran yang di-pengaruhi oleh waktu tunda. Tema yang dikedepankan adalah masalah kendali optimal H_2 untuk sistem tersebut. Ada dua solusi yang diberikan: solusi domein waktu dan solusi domein frekuensi.

Solusi domein waktu menawarkan formulasi yang sederhana dan mu-dah diterapkan. Pendekatan di domein ini menggunakan ide-ide dari pro-gram dinamik yang diadaptasi untuk sistem waktu tunda. Pengendali yang dihasilkan terdiri dari bagian dengan fungsi transfer rasional dan bagian dengan respon impuls hingga. Kedua bagian tersebut dapat dengan mudah diimplentasikan. Selain itu, bagian rasional dari pengendali mem-punyai orde yang sama dengan orde dari sistem. Satu-satunya kelemahan yang teramati dari pendekatan ini adalah bahwa metoda ini hanya dapat diterapkan jika waktu tunda hanya ditemui di salah sisi dari pengendali, di sisi masukan atau keluaran.

Pendekatan domein frekuensi menghasilkan sebuah solusi yang lebih umum, dalam arti bahwa solusi ini dapat diterapkan pada kasus dimana waktu tunda ditemui di kedua sisi pengendali. Pendekatan ini, yang di-dasari oleh teknik faktorisasi spektral, juga menghasilkan pengendali op-timal yang terdiri dari bagian rasional dan bagian respon impuls hingga. Hanya saja, formula-formula yang dihasilkan jauh lebih rumit dan kurang elegan dibanding formula-formula pada solusi domein waktu. Kelemah-an lain dari solusi domein frekuensi berkaitan dengan sistem tak stabil. Masalah kendali H_2 untuk sistem tak stabil tidak dapat diselesaikan meng-gunakan solusi domein frekuensi.

Pengembangan kedua solusi dari masalah kendali optimal H_2 seperti dibahas sebelumya memberikan efek samping berupa solusi dari parametrisasi Youla-Kučera untuk sistem dengan waktu tunda pada masukan/keluaran dan solusi dari masalah kendali optimal H_2 untuk sistem pra-pandang. Parametrisasi Youla-Kučera adalah sebuah formulasi yang menggambarkan semua pengendali yang menstabilkan sebuah sistem kendali secara internal. Selain merupakan sebuah hasil penelitian tersendiri, formulasi ini adalah bagian penting pada solusi domein frekuensi dari masalah kendali optimal H_2 untuk sistem dengan waktu tunda pada masukan/keluaran. Sistem pra-pandang adalah sistem dimana kita mempunyai informasi mengenai bagaimana bentuk (sebagian) sinyal masukan di masa depan. Solusi masalah kendali optimal H_2 untuk sistem demikian dapat diperoleh dengan menggunakan teknik yang sama dengan yang digunakan untuk menyelesaikan masalah tersebut untuk sistem dengan waktu tunda.

Acknowledgments

About the Author

Agoes Moelja was born on 18 April 1976 in Bandung, a city located in a mountainous region in the western part of the island of Java, Indonesia. He commenced his bachelor education at Institut Teknologi Bandung in 1994 and graduated with a bachelor degree in electrical engineering in August 1998. In 1999, he went to The Netherlands for a master study at University of Twente, where he obtained an M.Sc. degree in engineering mathematics in June 2001. In the same year, he joined the Signals, Systems, and Control Group, Department of Applied Mathematics at the same university as a Ph.D. student.